On properties of the American put option under several models

by

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Declarations

The work contained in this thesis is original, except as acknowledged, and has not been submitted previously for a degree at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made.

Chapter 3 is based on a joint paper with Dr Sigurd Assing. A version of which is available at arXiv:1411.6938.

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1 Introduction and Preliminaries

The central theme of this thesis is the application of optimal stopping in American option pricing. The purpose of this chapter is to provide a literature review of the techniques used throughout this thesis.

1.1 Optimal stopping problems

The theory of optimal stopping is concerned with the problem of choosing a time to take a particular action, in order to maximise an expected reward or minimise an expect cost. Optimal stopping theory has a long history. In 1947, Wald [75] investigated the problem of sequential testing. Snell [68] formulated a general optimal stopping problem for discrete time stochastic processes and characterised the solution by the means of smallest supermartingale dominating the gain sequence, which came to be known as Snell's envelope.

However, a general theory of the subject did not exist until the 1960s. In February 1960 copy of *Scientific American*, the famous Secretary Problem was discussed. The solution of which was suggested and proven by Dynkin [21]. In the 1963 paper, Dynkin also formulated a general optimal stopping problem for Markov processes and characterised the solution as the smallest superharmonic function dominating the gain function.

The 1960s and 70s saw major developments of general optimal stopping theory. [22, 23, 24, 25, 26, 29, 30, 40, 64, 65, 66] is a small sample of the extensive literature from this period. For a survey of optimal stopping problems and general theory of optimal stopping for Markov processes, [67] by Shiryaev is a good reference.

The classical applications of optimal stopping include mathematical statistics, stochastic analysis and financial mathematics. We refer to [57] and references within. The most relevant application of optimal stopping to this thesis is option pricing in the field of financial mathematics. The most famous example is the work on Black-Scholes model by McKean [49].

The purpose of this section is to review some of the standard techniques used in the existing literature for optimal stopping problems. We refer to [57, 67] for a comprehensive review. Some of the results we quote in this section are taken from these references.

We begin by defining optimal stopping problems in a standard set-up. In general, we are interested in optimal stopping problems with respect to strong Markov processes. Let $X = (X_t, \mathcal{F}_t : t \geq 0)$ be a strong Markov family defined on a measurable space $(\Omega, \mathcal{F})$ taking values in a measurable spaces $(E, \mathcal{B})$, together with a family of probability measure $\{\mathbb{P}_x\}_{x \in E}$. These probability measures have the property $\mathbb{P}_x(X_0 = x) = 1$ and $(X_t)_{t \geq 0}$ is a strong Markov process under $\mathbb{P}_x$. The filtration is taken to be the complete, augmented filtration generated by $X = (X_t, t \geq 0)$.

Intuitively, we can only make a decision about when to stop based on the information we already know. This is why the concepts of stopping time and Markov time are required.

Definition 1.1. *We define a Markov time to be a random variable τ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. A stopping time is a Markov time τ such that $\mathbb{P}(\tau < \infty) = 1$.*

For a given measurable function, $g : E \to \mathbb{R}$ which satisfies the condition

$$\mathbb{E}_x(\sup_{t\geq 0} |e^{-rt} g(X_t)|) < \infty \quad \text{ for all } x \in E, \tag{1.1}$$

we are interested in finding the value function

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x e^{-r\tau} g(X_\tau). \tag{1.2}$$

where $\mathcal{T}_{[0,\infty]}$ is the set of all stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ and r should be a positive constant. Here e^{-rt} represents a discount term, which can represent the opportunity cost for stopping later rather than sooner. The case $r \leq 0$ can lead to V being infinite and/or a stopping time does not exist, see Remark 2.2 for an example. This is referred to as an infinite horizon problem. Usually, we will drop $\mathcal{T}_{[0,\infty]}$ from our notation.

The corresponding finite time horizon problem is

$$V(x,t) = \sup_{0\leq \tau \leq T-t} \mathbb{E}_{t,x} e^{-r\tau} g(X_{t+\tau}), \tag{1.3}$$

This can be seen as replacing the X_t in equation (1.1) by the process $Z_t = (t, X_t)$ for $t \geq 0$ with the state space $\mathbb{R}_+ \times E$. By (1.3), we arrived at the **terminal condition** $V(x,T) = g(x)$ by setting $t = T$.

We refer to V as the **value function** and g as the **gain function**. The value function can be written in this simple form due to the Markov property of the process X. All of the processes we deal with in this thesis are Markov processes, so it is sufficient for our purposes to address the theory of optimal stopping in this form only. In addition to the value function, we are also interested in the stopping rule we need to apply to stop optimally, so we give the following definition:

Definition 1.2. *We say that τ^* is an optimal stopping time if*

$$V(x) = \mathbb{E}_x(e^{-r\tau^*} g(X_{\tau^*})). \tag{1.4}$$

Solving an optimal stopping problem consists of finding the pair (V, τ^*). We must have that $V(x) \geq g(x)$ by considering the stopping time $\tau = 0$. Therefore we may partition E

into two sets:

$$C = \{x \in E : V(x) > g(x)\}, \tag{1.5}$$

$$D = \{x \in E : V(x) = g(x)\}. \tag{1.6}$$

We refer to C as the continuation region and D as the stopping region. If V is lower semi-continuous and g is upper semi-continuous, then C is open and D is closed. Provided that the process X satisfies (1.1), then we have the following sufficiency theorem for optimal stopping.

Theorem 1.3 (Theorem 2.7 in [57]). *Assume that the gain function $g : E \to \mathbb{R}$ satisfying the condition*

$$E_x(\sup_{t\geq 0} e^{-rt}|g(X_t)|) < \infty \quad \text{for all } x \in E.$$

Assume that there exists a smallest function $\hat{V}$ which dominates the gain function g on E such that $e^{-rt}\hat{V}(X_t)$ is a supermartingale with respect to $\mathbb{P}_x$ for $x \in E$.

Let us in addition assume that $\hat{V}$ is lower semi-continuous (lsc) and g is upper semi-continuous (usc). Set $\hat{D} = \{x \in E : \hat{V}(x) = g(x)\}$ and let $\tau_{\hat{D}} = \inf\{t \geq 0 : X_t \in \hat{D}\}$. We then have:

(1) if $\mathbb{P}_x(\tau_{\hat{D}} < \infty) = 1$ for $x \in E$, then $\hat{V} = V$ and $\tau_{\hat{D}}$ is optimal in (1.2),

(2) if $\mathbb{P}_x(\tau_{\hat{D}} < \infty) < 1$ for some $x \in E$, then there is no optimal stopping time (with probability 1).

In the case when V is lsc, this is equivalent to the statement $V(x)$ is an r**-excessive function**. It turns out this r-excessive function characterisation is not only a sufficient condition the value function needs to satisfy, but also a necessary condition. Let us define τ_D by

$$\tau_D = \inf\{t \geq 0 : X_t \in D\},$$

which is a Markov time with respect to $(\mathcal{F}_t)_{t\geq 0}$. We have the following necessary condition for optimal stopping.

Theorem 1.4 (Theorem 2.4 in [57]). *Let us assume that there exists a stopping time τ^* such that (1.4) holds for all $x \in E$, then*

(1) $e^{-rt}V(X_t)$ is the smallest right-continuous supermartingale under $\mathbb{P}_x$ such that $V(x) \geq g(x)$ for all $x \in E$.

In addition, if V is lsc and g is usc, then we have:

(2) The stopping time $\tau_D \leq \tau^*$ $\mathbb{P}_s$-a.s. for all $x \in E$ and is optimal in (1.4).

(3) *The stopped process* $\{e^{-r(t\wedge\tau_D)}V(X_{t\wedge\tau_D}) : t \geq 0\}$ *is a right-continuous martingale under* $\mathbb{P}_x$ *for every* $x \in E$.

Remark 1.5. It is useful to make the following observations.

(i) Theorem 1.3 and Theorem 1.4 also hold for finite horizon problem where we would replace $V(X_t)$ by $V(X_t, t)$.

(ii) In finite time horizon problems, due to the terminal condition $V(T, x) = g(x)$, τ_D is an optimal stopping time with $\tau_D \leq T$.

In infinite horizon problems, there may not exist an optimal stopping time τ^* at which the value function V is attained. However, by the definition of V, for all $\epsilon > 0$, there exists a stopping time τ^*_ϵ such that

$$V(x) - \epsilon \leq \mathbb{E}_x(e^{-r\tau^*_\epsilon} g(X_{\tau^*_\epsilon})) \tag{1.7}$$

A stopping time satisfying equation (1.7) is known as an ϵ-optimal stopping time.

(iii) When an optimal stopping time does not exists, sometimes we can find Markov time τ^* such that (1.4) holds. If τ_D is a Markov time instead of a stopping time and $\mathbb{P}(\lim_{t\to\infty} e^{-rt}g(X_t) \to 0) = 1$, then (1.4) still holds with a Markov time $\tau^* = \tau_D$ with the convention $e^{-r\tau^*}g(X_{\tau^*}) = 0$ when $\tau^* = \infty$. This is the case in some of the problems we will discuss in Chapter 2-4.

(iv) When we cannot solve the optimal stopping problem explicitly, we are still interested in the nature of the continuation region and its boundary, as this allows us to characterise the optimal stopping rule. Literature in this area include [37, 20].

An implication of Theorem 1.3 and Theorem 1.4 is that solving an optimal stopping problem (1.4) is equivalent to finding the smallest r-excessive function $\hat{V}$ such that $\hat{V} \geq g$. In this case, $V(x) = \hat{V}(x)$ for $x \in E$ and $\hat{\tau}_D$ defined in Theorem 1.3 is an optimal stopping time if $P_x(\tau_{\hat{D}} < \infty) = 1$ for all $x \in E$.

Generally, it can be quite difficult to search for and identify the smallest r-excessive function directly hence Theorem 1.3 is often not a practical method for solving an optimal stopping problem. An alternative approach is known as the guess and verify approach.

1.1.1 Guess and Verify

Instead of trying to identify the smallest r-excessive function, we can try to identify an optimal stopping rule. The stopping time associated with this rule τ would produce a function $\hat{V}$ which corresponds to the expected discount gain function when stopping according this rule. We then verify that $\hat{V}$ is indeed the value function.

The result below is often referred to as the guess and verify lemma. See [1, 11, 53, 71] for examples of its application.

Lemma 1.6. *Consider the optimal stopping problem given by (1.2) with $g \geq 0$. Let τ be a stopping time and $\hat{V} : E \to \mathbb{R}$ be a function such that the following three conditions are satisfied*

(i) $\hat{V}(x) = \mathbb{E}_x e^{-r\tau} g(X_\tau)$,

(ii) $\hat{V}(x) \geq g(x)$ *for all* $x \in E$,

(iii) the process $\{e^{-rt}\hat{V}(X_t) : t \geq 0\}$ *is a right-continuous supermartingale under* $\mathbb{P}_x$ *for all* $x \in E$,

then $V = \hat{V}$ *and* τ *is an optimal stopping time.*

One advantage of this approach is that some properties of the value function does not need to be proved. For example, we often expect the value function to solve a free-boundary problem with a smooth pasting condition. By imposing a smooth pasting conditioning and verifying that the solution of the free-boundary is the value function, we do not need to prove the pasting condition. This lemma is used extensively in Chapter 3.

On the other hand, it is sometimes difficult to guess the value function. Even if we guessed the stopping rule correctly (or proved that it must be of a certain form), the verification procedure can be difficult. This problem is highlighted in Chapter 2 for the Regime Switching model. This method, in general, does not work in finite time horizon because of the stopping region depends on time. It is generally difficult to compute the expected value of the gain function at bounded stopping times.

1.1.2 Free-boundary approach

One approach in solving optimal stopping problems is by solving a free boundary problem. Let us assume that X solves an SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{B}_t \tag{1.8}$$

and $c \in \mathbb{R}$. If we assume that V is sufficiently regular, then by Itô's formula, we have that

$$\begin{aligned} \mathbb{E}_x e^{-rt} V(X_t) =& V(x) + \mathbb{E}_x\bigg(\int_0^t (L - r)V(X_u)\mathbf{1}_{\{X_s \neq c\}}\mathrm{d}u \\ &+ \frac{1}{2}\int_0^t (V'(X_s+) - V'(X_s-))\mathbf{1}_{\{X_s = c\}}\mathrm{d}l_s^c(X) + M_t\bigg), \end{aligned} \tag{1.9}$$

where L is the infinitesimal generator of X, $l_s^c(X)$ is the local time of X at c and M_t is a local martingale with $M_0 = 0$.

If we know the stopping region D and the continuation region C given by (1.5) and (1.6) are characterised by some value of c, for example, if

$$C = \{x : x > c\},$$

then it is nature to guess that V is the unique solution the problem

$$(L - r)V(x) \le 0 \qquad \text{for } x \neq c \tag{1.10}$$

$$(L - r)V(x) = 0 \qquad \text{for } x \in C \tag{1.11}$$

$$V(x) = g(x) \qquad \text{for } x \in D \tag{1.12}$$

$$V(x) \ge g(x) \qquad \text{for } x \in \mathbb{R} \tag{1.13}$$

$$V'(c) = g'(c) \tag{1.14}$$

and M_t is a martingale. It can often be verified by Lemma 1.6 that the unique solution to (1.10) to (1.14) is the value function. A modified version of this Ansatz exists when X is multidmensional, has jumps or the problem is in finite horizon. This is known as a free-boundary problem because the boundary c is unknown.

Some optimal stopping problems can be solved explicitly by finding the unique solution to their related free-boundary problems. A classical example of this is the McKean problem [49]. In higher dimension, it may be possible to derive a Volterra type integral equation from the free-boundary problem. One needs to verify that the unique solution of integral equation is the value function. See [57, Theorem 25.3] for how this is done for the finite horizon Black-Scholes American put problem. The condition $V'(c) = g'(c)$ is often known as a 'smooth pasting' condition. This condition first appeared in papers by Mikalevich [52] and [51]. Other works on smooth pasting include [14, 10, 49, 36].

The potential difficulty in using the free-boundary approach is two-fold. Firstly, it is necessary to show the uniqueness of the solution to the free-boundary problem. For the finite horizon American put problem under Black-Scholes model, the uniqueness of solution to the integral equation was not proved until 2005 in [56].

Secondly, it is often difficult to verify that the (unique) solution to the free-boundary problem is indeed the solution to the optimal stopping problem. This problem is discussed extensively for the Regime-Switching model in Section 2.1.3. This is why the notion of viscosity solution introduced in the next section comes in handy.

1.2 Viscosity Solutions

The notion of viscosity solutions was introduced by Crandall and Lions [18] in the 1980s for Hamilton-Jacobi equations related to control problems. It turns out that the notion of viscosity solution can be used for studying optimal stopping problems. Consider the variational inequality

$$\min(-(L-r)V(x),\ V(x)-g(x))=0. \tag{1.15}$$

This is an alternative way to write the free boundary problem given by equations (1.10) to (1.14). One can check that if a function satisfies the free boundary problem in the classical sense, then it also satisfies (1.15), except at the stopping boundary c. Viscosity solutions provide us with a weaker notion of solution which is consistent with the classical definition and allows the value function of optimal stopping problem (1.2) to be a solution of this equation (1.15).

The theory of viscosity solutions was initially developed only for first order PDEs, but this was soon extended to integral partial differential equations (abbreviated as IPDEs or PIDEs). These are partial differential equation operators with non-local parts. One of the first papers on viscosity solutions for PIDEs was [70] by Soner, which extended the viscosity solution framework to piecewise-deterministic jump processes with bounded coefficients. This work was then extended by Sayah in [63].

It was initially unknown whether second order elliptic equations admit unique viscosity solutions in general. The breakthrough came in 1988, when Jensen proved a comparison principle in [38]. For a comprehensive account of viscosity solution for degenerate ellipitic second order partial differential equations, we refer to the User's guide [19]. For papers related to second order degenerate elliptic PIDE, we refer to [4, 5] and references within. The first applications of viscosity solution in optimal stopping include [55] and [59].

The theory of viscosity solution applies to PDEs of the form

$$F(x,u,Du,D^2u)=0, \tag{1.16}$$

where $F:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times S(N)\to\mathbb{R}$ is continuous and $S(N)$ is the set of symmetric $N\times N$ matrices. Du and D^2u denote the gradient and the second derivative matrix of u.

Furthermore, F is required to satisfy the fundamental assumption,

$$F(x,r,p,X)\le F(x,s,p,Y) \text{ whenever } r\le s \text{ and } Y\le X. \tag{1.17}$$

where $r,s\in\mathbb{R}, x,p\in\mathbb{R}^N, X,Y\in S(N)$. If (1.17) is satisfied by F, then F is said to be proper.

It can be easily verified that the variational inequality (1.15) satisfies condition (1.17), if L is the generator of one dimensional diffusion. The same restriction applies to the differential part of PIDEs operators related to wide classes of jump processes. In the case of PIDEs, some care needs to be taken in dealing with the integral part. We are now ready to introduce the notion of viscosity solution.

Definition 1.7. *Let F satisfy (1.17) and $\mathcal{O} \subset \mathbb{R}^N$. A continuous viscosity subsolution of $F = 0$ on $\mathcal{O}$ is a continuous function u such that for every C^2 function ϕ such that $\hat{x}$ is a local maximum of $u - \phi$, then*

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(x)) \leq 0.$$

Similarly, a continuous viscosity supersolution of $F = 0$ is a continuous function u such that for every C^2 function ϕ such that $\hat{x}$ is a local minimum of $u - \phi$, then

$$F(\hat{x}, u(\hat{x}), D\phi(\hat{x}), D^2\phi(x)) \geq 0.$$

u is a solution if it is both a subsolution and a supersolution

An equivalent formulation of viscosity solution property is in terms of semijets. The superjets of u at x is given by

$$J^{2,+}_{\mathcal{O}}u(x) = \left\{(p, X) \in \mathbb{R}^N \times S(N) : u(\hat{x}) \leq u(x) + p(x - x') + \frac{1}{2}(x' - x)^T X(x' - x) + o\big(|x - x'|^2\big)\big)\right\}.$$

the subjets of u at x is defined by

$$J^{2,-}_{\mathcal{O}}u(x) = -J^{2,+}_{\mathcal{O}}(-u(x)).$$

It is possible to show that for all $(p, X) \in J^{2,+}_{\mathcal{O}}u(x)$, there exists a C^2 function ϕ such that $u - \phi$ has a local maximum at x with $D\phi(x) = p$ and $D^2\phi(x) = X$. Hence, an equivalent definition of viscosity solution is in term of these semijets.

Definition 1.8. *Let F satisfy (1.17) and $\mathcal{O} \subset \mathbb{R}^N$. A continuous viscosity subsolution of $F = 0$ on $\mathcal{O}$ is a continuous function u such that*

$$F(x, u(x), p, X) \leq 0 \quad \textit{for all } x \in \mathcal{O} \textit{ and } (p, X) \in J^{2,+}_{\mathcal{O}}u(x).$$

Similarly, a continuous viscosity subsolution of $F = 0$ is a continuous function u such that

$$F(x, u(x), p, X) \geq 0 \quad \textit{for all } x \in \mathcal{O} \textit{ and } (p, X) \in J^{2,-}_{\mathcal{O}}u(x).$$

We have now defined the notion of the viscosity solution. It is possible to show that the value function is the viscosity solution to a variational inequality.

Theorem 1.9 (Theorem 5.2.1 of [60]). *Recall the definition of V given by (1.2), where the gain function g is now assumed to be Lipschitz. Assuming that r is sufficiently large, then V is the unique viscosity solution to (1.15) satisfying a linear growth condition.*

Similar results exist if the optimal stopping problem is a finite time horizon one or the underlying process has jumps. As a consequence, the definitions of the subsolution and supersolution need to be adjusted accordingly. We will introduce case specific definitions of supersolution and subsolution in Chapter 2 and 4. Since the notion of viscosity solution is consistent with the classical definition of solution of a PDE, we can apply regularisation arguments to prove properties of the value function without solving for the value function. The regularisation arguments in Chapter 4 will need uniqueness theorem of the following type.

Theorem 1.10 (Theorem 3.3 of [19])**.** *Let Ω be a bounded open set in $\mathbb{R}^N$, $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N, S(N))$ be proper and satisfy the conditions*

(i) There exists $\gamma > 0$ such that

$$\gamma(r-s) \leq F(x,r,p,X) - F(x,s,p,X) \text{ for } r \geq s$$

(ii) There is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that

$$\begin{cases} F(y,r,\alpha(x-y),Y) - F(x,r,\alpha(x-y),X) \leq \omega(\alpha|x-y|^2 + |x-y|) \\ \text{whenever } x,y \in \Omega, r \in \mathbb{R}, X,Y \in S(N), \text{ and } X,Y \text{ satisfy the equation} \end{cases}$$

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

Let u be a continuous subsolution on Ω and v be a continuous supersolution of $F = 0$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in $\bar{\Omega}$.

1.3 Outline of this thesis

The content of the rest of this thesis is outlined below.

Chapter 2 This chapter is dedicated to the Regime Switching model. In the first part of chapter, we study optimal stopping problems under the Regime-Switching model using the notion of viscosity solution. We correct some of the discrepancies in the existing literatures about the perpetual American put problem. In the second part of this chapter, we strengthen an existing result on the monotonicity property of option price in volatility and discuss its implication for numerical schemes.

Chapter 3 In this chapter, we propose a new type of stochastic volatility model, where a volatility change occurs upon the price process hitting a critical level. We call this model an interactive volatility model. We consider the problem of trading American put options under this model. It turns out that the value function and the optimal stopping rules under model can be found explicitly. For certain choice of parameters, it has an interesting feature that the stopping region is disconnected. We end this chapter with a numerical analysis section with discussions.

Chapter 4 In this chapter, we study optimal stopping problems under the BNS model, which is a stochastic volatility model with jumps. This model poses additional difficulty in comparison to jump diffusion models and stochastic volatility models without jumps. We use a combination of probabilistic and analytical tools to prove a number of monotonicity and regularity results for the value function and the stopping boundary/surface. At the end of this chapter, we produce some numerical estimates of the option prices using a Monte Carlo approach.

2 On Optimal Stopping under Regime-Switching and Related Models

2.1 Introduction

2.1.1 Literature review and chapter summary

As noted by many authors, the Black-Scholes model despite being very successful, does not have many desired properties of a market model. One relatively simple attempt to add extra randomness to the model is to let the volatility and rate of return be functions of a finite state Markov chain.

Under the Regime-Switching model, we take a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports two independent Markov processes $(W_t)_{t\geq 0}$ and $(I_t)_{t\geq 0}$, where W is a Brownian motion and I is an n state continuous time Markov chain with state space $\{1, \ldots, n\}$. Let S_t be driven by the stochastic differential equation

$$\mathrm{d}S_t = \sigma(I_t) S_t \mathrm{d}W_t + \mu(I_t) S_t \mathrm{d}t, \tag{2.1}$$

where $\sigma : \{1, \ldots, n\} \to (0, \infty)$ and $\mu : \{1, \ldots, n\} \to (-\infty, \infty)$ are known functions. The S_t governed by (2.1) is referred to as a process with 'Regime-Switching' or a 'Markov modulated geometric Brownian motion' by the existing literature. Intuitively, the Markov chain can be regarded as modelling the business cycle or some economic indicator. It is straightforward to see, when the functions μ and σ are chosen to be constant functions, or $n = 1$, then this model reduces to the Black-Scholes model.

This model has been subject to intense studies. From an option pricing perspective, see Guo [32] for closed-form solutions for pricing European and perpetual lookback option; Yao, Zhang and Zhou [76] for numerical results for European stock options. Two papers on pricing perpetual American puts are particularly relevant to this chapter: one by Guo and Zhang, [33], treating the case of two-states Markov chains and another by Jobert and Rogers, [39], treating the general case with finitely many states. In addition, the work by Buffington and Elliot, [12], studied the two-state finite horizon American put problem.

This chapter has two parts. First, we introduce the probabilistic set-up for the Regime-Switching model and present the existing results in [33] and [39]. We are unable to follow some of the arguments in [33, 39]. It appears that the proofs of the main results in these papers are incomplete. We examine these in detail and demonstrate how the problems in [33, 39] can be addressed using viscosity solutions.

In the second part of this chapter, we address the question, under what conditions are option prices monotone in volatility. We generalise a result from [2] and prove two new monotonicity theorems. We compare these results and apply them to American put options. We demonstrate that our results on American put options are consistent with existing results on the Black-Scholes model by McKean [49] as well as numerical results found in [15] and [42]. We end this chapter with a discussion of open problems and some conjectures based on our results.

2.1.2 Probabilistic set-up for the Regime-Switching model

We recall the probabilistic set-up given in the paragraph preceding equation (2.1). The Regime-Switching process S_t is governed by

$$\mathrm{d}S_t = \mu(I_t)S_t\mathrm{d}t + \sigma(I_t)S_t\mathrm{d}W_t,$$

and adapted to $\mathcal{F}_t$ which we take to be the complete augmented filtration generated by $\sigma((W_s, I_s) : s \le t)$. We denote the generator matrix of I by Λ and use λ_{ij} denote its entries.

This means, for $i \neq j$, λ_{ij} is the jump rate from state i to state j. For $i = j$, λ_{ii} is the total rate leaving state i multiplied by -1. In addition, denote the total rate leaving i by λ_i, i.e.

$$\lambda_i \stackrel{\text{def}}{=} -\lambda_{ii}.$$

We construct a family of these processes with initial values $S_0 > 0$ and $I_0 \in \{1, \ldots, n\}$. We use $S^{s,i}$ to denote the process S with $S_0 = s$ and $I_0 = i$. Similarly, we use I^i to denote the process I with $I_0 = i$. (The distribution of future value of I does not depend on current or past values of S, when conditioned on its current value.) $S_t^{s,i}$ has the explicit representation

$$S_t^{s,i} = s\exp\left(\int_0^t \mu(I_q^i) - \frac{1}{2}\sigma(I_q^i)^2\mathrm{d}q + \int_0^t \sigma(I_q^i)\mathrm{d}W_q\right). \tag{2.2}$$

It is also useful to define $X_t^{x,i} = \log S_t^{s,i}$, where $x = \log s$. Explicitly, we have that

$$X_t^{x,i} = x + \int_0^t \mu(I_q^i) - \frac{1}{2}\sigma(I_q^i)^2\mathrm{d}q + \int_0^t \sigma(I_q^i)\mathrm{d}W_q.$$

The pairs $(S^{s,i}, I^i)$ and $(X^{x,i}, I^i)$ are both Markov processes. Let g and h be real functions such that $g : (0,\infty) \times \{1, \ldots, n\} \to \mathbb{R}$ and $h : (0,\infty) \times \{1, \ldots, n\} \to \mathbb{R}$. We are interested in the value function associated with the optimal stopping problem

$$u(s,i) = \sup_\tau \mathbb{E}\left[e^{-\int_0^\tau r(I_t^i)\mathrm{d}t} g(S_\tau^{s,i}, I_\tau^i) + \int_0^\tau e^{-r(I_t^i)t} h(S_t^{s,i}, I_t^i)\mathrm{d}t\right] \quad \text{for } s > 0,\ i \in \{1, \ldots, n\}, \tag{2.3}$$

where $r : \{1, \ldots, n\} \to (0, \infty)$ and the supremum is taken over the class of all stopping times with respect to $\mathcal{F}_t$. This problem also has the representation

$$\tilde{u}(x,i) = \sup_{\tau} \mathbb{E}\Big[e^{-\int_0^\tau r(I_t^i)\mathrm{d}t}\tilde{g}(X_\tau^{x,i}, I_\tau^i) + \int_0^\tau e^{-r(I_t^i)t}\tilde{h}(X_t^{x,i}, I_t^i)\mathrm{d}t\Big], \tag{2.4}$$

where $\tilde{g}, g, \tilde{h}, h, \tilde{u}, u$ are related by $\tilde{g}(x,i) = g(e^x, i)$, $\tilde{h}(x,i) = h(e^x, i)$, $\tilde{u}(x,i) = u(e^x, i)$. We assume that there is an $r < \min_{i=1}^n r(i)$ such that

$$\mathbb{E}\sup_{t\geq 0} e^{-rt} g(S_t^{s,i}, I_t^i) < \infty \quad \text{and} \quad \mathbb{E}\sup_{t\geq 0}\int_0^t e^{-ru} h(S_u^{s,i}, I_u^i)\mathrm{d}u < \infty \tag{2.5}$$

are satisfied, hence $u(s,i)$ is finite. We define the regions C_i and D_i for $i = 1, \ldots, n$ as follows:

$$C_i = \{s : u(s,i) > g(s,i)\}, \quad D_i = \{s : u(s,i) = g(s,i)\}. \tag{2.6}$$

It is straightforward to see that the C_i's and the $D_i's$ partition the state space into continuation regions and stopping regions according to the value of i. Similarly, define

$$\tilde{C}_i = \{x : \tilde{u}(x,i) > \tilde{g}(x,i)\}, \quad \tilde{D}_i = \{x : \tilde{u}(x,i) = \tilde{g}(x,i)\}.$$

In what follows, we shall use the notation $\sigma_i = \sigma(i)$, $\mu_i = \mu(i)$ and $r_i = r(i)$ for $i \in \{1, \ldots, n\}$. If $f : (0,\infty) \times \{1, \ldots n\} \to \mathbb{R}$ is a function such that $f(\cdot, i) \in C_b^2(0,\infty)$ for every i, then the infinitesimal generator of the pair (S, I) acting on f is given by

$$Lf(s,i) = \frac{1}{2}\sigma_i^2 s^2 \partial_{11} f(s,i) + \mu_i s \partial_1 f(s,i) - \lambda_i f(s,i) + \sum_{j\neq i} \lambda_{ij} f(s,j).$$

Similarly, for a function $f : (-\infty,\infty) \times \{1, \ldots, n\} \to \mathbb{R}$ such that $f(\cdot, i) \in C_b^2(-\infty,\infty)$ for every i, the infinitesimal generator of the pair (X, I) acting on f is given by

$$\tilde{L}f(x,i) = \frac{1}{2}\sigma_i^2 \partial_{11} f(x,i) + (\mu_i - \frac{1}{2}\sigma_i^2)\partial_1 f(x,i) - \lambda_i f(x,i) + \sum_{j\neq i} \lambda_{ij} f(x,j).$$

Remark 2.1. The most relevant case to option pricing is the case $\mu_i = r_i$, because this is the necessary condition to ensure the model has no arbitrage. However, the other cases can be interesting to a market participant who has his own view about what these parameters actually are. He may wish to know the value of the option that is consistent with his view of the market.

2.1.3 Existing results on the American put problem under the Regime-Switching model

The American put is a case of (2.3) where

$$g(s,i) = (K-s)^+ \quad h(s,i) = 0 \qquad \text{for } s > 0,\ i = \{1,\dots,n\}.$$

It is claimed in [39, Prop. 2] that, for $\mu_i \le r_i$, if thresholds $\tilde{b}_i < \log(K)$ have been found such that the unique bounded solution $\tilde{f}$ to the coupled system of ODEs

$$\begin{aligned} (\tilde{L} - r_i)\tilde{f}(x,i) &= 0 && \text{for } x > \tilde{b}_i, \\ \tilde{f}(x,i) &= K - e^x && \text{for } x \le \tilde{b}_i, \end{aligned}$$

is C^1 in x at $(\tilde{b}_i, i)$ for every i, then the $\tilde{b}_i$'s are uniquely determined and $\tilde{u}(x,i) = \tilde{f}(x,i)$. If this result holds, by a simple change of variable, we have that $u(s,i)$ is the unique bounded function satisfying

$$(L - r_i)f(s,i) = 0 \qquad \text{for } s > b_i, \tag{2.7}$$

$$f(s,i) = K - s \qquad \text{for } s \le b_i. \tag{2.8}$$

such that $f(\cdot, i)$ is C^1. Here, $b_i = \exp(\tilde{b}_i)$, is the stopping level for S when I is in state i. The result is stated in [33] in term of f for the case $n = 2$, $\mu_i \ge 0$ and $r_1 = r_2 > 0$. In the two-states case, the solution is semi-explicit.

Attempts were made in [39, 33] to check that the solution to system of coupled ODEs are indeed the value functions. These attempts used 'guess and verify' methods. If we want to check that $f = u$ by Lemma 1.6, we need to show the following conditions hold:

(v1) $f(s,i) = \mathbb{E}[e^{-\int_0^\tau r(I_q^i)\mathrm{d}q} g(S_\tau^{s,i})]$

(v2) $e^{-\int_0^t r(I_q^i)\mathrm{d}q} g(S_t^{s,i})$ is a supermartingale for all $s > 0$, $i \in \{1,\dots,n\}$.

(v3) $f(s,i) \ge (K-s)^+$.

In [39, Prop. 1], the authors claim that the stopping regions D_i must be of the form $D_i = (0, b_i)$ assuming $\mu_i = r_i$. However, their argument only requires $S_t^{s,i} e^{-\int_0^t r(I_q^i)\mathrm{d}q}$ to be a supermartingale, which holds if $\mu_i \le r_i$.

Remark 2.2. [39, Prop. 1] seems to hold even if $r_i \le 0$. We briefly comment on the consequences when the discount term is non-positive. When $r_i \le 0$, $\mu_i < \frac{1}{2}\sigma_i^2$, $X_t^{x,i} \to -\infty$ almost surely. For any $\eta < K$, consider the stopping time $\tau_\eta^{s,i} = \inf\{t : S_t^{s,i} \le \eta\}$, then

$$u(s,i) \ge \mathbb{E}e^{-r\tau_\eta^{s,i}}(K - S_{\tau_\eta^{s,i}}^{s,i})^+ = \mathbb{E}e^{-r\tau_\eta^{s,i}}(K - \min(s,\eta)) \ge (K - \min(s,\eta))(1 + r\mathbb{E}\tau_\eta^{s,i}).$$

Since η is arbitrary, we have that

$$u(s,i) = \begin{cases} \infty & \text{for } r < 0 \\ K & \text{for } r = 0. \end{cases}$$

In these cases, an optimal stopping time (or Markov time) does not exist.

When examined more closely, it looks to us that they assumed that an optimal stopping time exists and applied Doob's optional sampling theorem to an unbounded martingale with this stopping time. It turns out [39, Prop. 1] can be proved using a convexity argument provided we assume $r_i > 0$, see Lemma 2.9 on page 24. With this in mind, we examined [39, Prop. 2]. In this proposition, the authors in particular demonstrate $(\tilde{L} - r_i)\tilde{f}(x,i) \leq 0$ when $\mu_i = r_i$. We cannot follow their proof of (v3) as it appears that they again applied Doob's optional sampling theorem using an unbounded stopping time. Furthermore, we do not see why, without using further arguments, the non-linear equations mentioned in [39, Problem 2] should have unique solutions. See Remark 2.3 for an example where this does not hold for a general convex function, even in the case when $n = 1$.

Following [33], one realises that the authors do not verify but assume (v3) in their Theorem 3.1. Furthermore, their proof of (v2) is incomplete as they only justify $(L - r_i)f(s,i) \leq 0$ inside of the continuation region. Outside the continuation region, that is, when $u(\cdot, i)$ coincides with $(K - \cdot)^+$, the validity of $(L - r_i)f(s,i) \leq 0$ would depend on the value of b_i, and this issue was not addressed in [33].

In addition, it is claimed in [33, 39] that the solution of the free-boundary value problem is unique and equal to the value function. Neither existence nor uniqueness part is clear to us, hence it is not clear that a solution of (2.7) and (2.8) (if exists) is the actual value function. In this section, we take a different approach. We first show that the stopping region of the American put option is characterised by stopping levels b_i, then we show that (2.7) and (2.8) are <u>necessary</u> conditions for the value function u.

Remark 2.3. Uniqueness is not an intrinsic property of solutions of free-boundary value problem. Here is an example where the free-boundary problem admits more than one solution.

Let $g : (0, \infty) \to \mathbb{R}$ be a bounded smooth convex function satisfying

$$g(x) = \left\{ \begin{array}{ccc} 4 & : & x = 1/2 \\ 1 & : & x = 1 \end{array} \right. , \quad g'(x) = \left\{ \begin{array}{ccc} -8 & : & x = 1/2 \\ -1 & : & x = 1 \end{array} \right. , \quad g(x) = 0, \, x \geq 3,$$

and consider the value function $V(x) = \sup_{\tau \geq 0} \mathbb{E}[e^{-\tau} g(X^x_\tau)]$ where $X^x_t = xe^{\sqrt{2}B_t}$, $t \geq 0$, is a geometric Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Following the method used in [33, 39], the free-boundary value problem associated with this optimal stopping problem is

$$0 \;=\; xV'(x) + x^2V''(x) - V(x) \quad \text{for} \quad x > x_0$$

subject to

$$V(x_0) = g(x_0), \quad V'(x_0) = g'(x_0), \quad \lim_{x\to\infty} V(x) = 0.$$

As any solution to this problem must have the form $c_1x + c_2x^{-1}$, the above boundary and pasting conditions result in $c_1 = 0$ and two equations

$$g(x_0) \;=\; c_2x_0^{-1}, \quad g'(x_0) \;=\; -c_2x_0^{-2}$$

for the pair of unknowns (c_2, x_0).

There are at least two solutions to these equations, $(c_2, x_0) = (1, 1)$ and $(c_2, x_0) = (2, 1/2)$, but there might be even more. Note that the value function is unique and can only be identical to one of the candidate value functions build from these solutions (c_2, x_0).

2.2 Regularity properties of perpetual American put problem under Regime-Switching model

We introduce the relevant variational inequality for (2.3). Consider the equation

$$\min(-\tilde{L}\tilde{f}(x,i) - \tilde{h}(x,i) + r_i\tilde{f}(x,i),\ \tilde{f}(x,i) - \tilde{g}(x,i)) = 0. \tag{2.9}$$

We say $\psi \in \mathcal{W}$ if $\psi : \mathbb{R} \times \{1, \ldots, n\} \to \mathbb{R}$ and $\psi(\cdot, i)$ is C^2 for every $i \in \{1, \ldots, n\}$. A function $w : \mathbb{R} \times \{1, \ldots, n\} \to \mathbb{R}$ is a viscosity subsolution (supersolution) of (2.9) if for every $\psi \in \mathcal{W}$ such that

(i) $\psi(x, i) = w(x, i)$ for some $x \in \mathbb{R}$ and all $i \in \{1, \ldots, n\}$,

(ii) $\psi(x', i') \geq w(x', i')$ $(\leq)$ for all $(x', i') \in \mathbb{R} \times \{1, \ldots, n\}$,

ψ also satisfies

$$\min(-\tilde{L}\psi(x,i) - \tilde{h}(x,i) + r_iw(x,i),\ w(x,i) - \tilde{g}(x,i)) \geq 0 \ (\leq). \tag{2.10}$$

A function w is a solution if it is both a subsolution and a supersolution. The following theorem is a summary of Theorem 2.1 and Lemma 2.3 of [78]. The proof of this theorem is a straightforward adaptation of [60, Theorem 5.2.1]

Theorem 2.4. *Let $\tilde{g}(\cdot, i)$ and $\tilde{h}(\cdot, i)$ be Lipschitz continuous for every $i \in \{1, \ldots, n\}$. Assume that r_i's are sufficiently large, then* *(i)* $\tilde{u}(\cdot, i)$ *is Lipschitz for every* $i \in \{1, \ldots, n\}$,

(ii) $\tilde{u}(x, i)$ *is the unique viscosity solution with at most linear growth to the variational equation (2.10).*

Remark 2.5. The restriction on the value of r_i is only needed to ensure $\tilde{u}$ is Lipschitz. In the case of the Regime Switching model, $|X_t^{x,i} - X_t^{x',i}| = |x - x'|$. It follows that, if $r = \min_{i=1}^n r_i > 0$, then

$$
\begin{aligned}
&|\tilde{u}(x,i) - \tilde{u}(x', i)| \\
&\quad = \sup_\tau \mathbb{E}\left[\int_0^\tau |\tilde{h}(X_t^{x,i}, I_t^i) - \tilde{h}(X_t^{x',i}, I_t^i)|\mathrm{d}t + e^{-\int_0^\tau r(I_t^i)\mathrm{d}t}|\tilde{g}(X_t^{x,i}, I_t^i) - \tilde{g}(X_t^{x',i}, I_t^i)|\right] \\
&\quad \leq C\mathbb{E}\left[\int_0^\infty e^{-r(I_t^i)t}|x - x'|\mathrm{d}t\right] + C\mathbb{E}\sup_{t\geq 0}|X_t^{x,i} - X_t^{x',i}| \\
&\quad \leq C\mathbb{E}\left[\int_0^\infty e^{-rt}|x - x'|\mathrm{d}t\right] + C\mathbb{E}\sup_{t\geq 0}|X_t^{x,i} - X_t^{x',i}| \\
&\quad \leq C'|x - x'|,
\end{aligned}
$$

where C is a Lipschitz constant for both $\tilde{g}$ and $\tilde{h}$. C' is another constant.

We now prove that the value function is smooth in the continuation region. This is relatively standard and there are a number of ways to do this. The method we present here appeals to the following theorem in one dimension.

Proposition 2.6 (Proposition 5.2.1 of [60])**.** *Let g and h be Lipschitz continuous functions from $\mathbb{R}$ to $\mathbb{R}$, X a one-dimensional diffusion driven by the SDE*

$$dX_t = b(X)dt + \sigma(X_t)dW_t,$$

with X_t^x denoting the process with $X_0 = x$. Assume that b and σ satisfies the usual Lipschitz conditions ensuring the existence and uniqueness of a solution. If σ is bounded away from 0, then the value function w defined by

$$w(x) = \mathbb{E}\left[e^{-r\tau}g(X_\tau^x) + \int_0^\tau e^{-rt}h(X_t^x)\,dt\right]$$

is C^2 in the continuation region C. Moreover, w is C^1 on ∂C if g is C^1 on ∂C.

Proposition 2.7. *Recall the definition of C_i given by (2.6) on page 18. Assuming that $r_i > 0$, $u(\cdot, i)$ is C^2 in C_i. Moreover, u is C^1 at ∂C_i if g is C^1 at ∂C_i for all $i \in \{1, \ldots, n\}$.*

Proof. Fix an i and define $\tilde{X}$ to be a family of Markov process with initial condition $\tilde{X} = x$

defined on $(\Omega, \mathcal{F}, \mathbb{P})$. $\tilde{X}$ is driven by the stochastic differential equation

$$\mathrm{d}\tilde{X}_t = \sigma_i \mathrm{d}W_t + (\mu_i - \tfrac{1}{2}\sigma_i^2)\mathrm{d}t. \tag{2.11}$$

The generator of $\tilde{X}$ is given by

$$\hat{L}f(x) = \tfrac{1}{2}\sigma_i^2 f''(x) + (\mu_i - \tfrac{1}{2}\sigma_i^2)f'(x).$$

Now consider the optimal stopping problem

$$w(x) = \sup_\tau \mathbb{E}\bigg[e^{-(r_i+\lambda_i)\tau}\tilde{g}(\tilde{X}_\tau^{x,i}, i) + \int_0^\tau e^{-(r_i+\lambda_i)t}\Big(\tilde{h}(\tilde{X}_t^{x,i}, i) + \sum_{j\neq i}\lambda_{ij}\tilde{u}(\tilde{X}_t^{x,i}, j)\Big)\mathrm{d}t\bigg].$$

By Theorem 2.4 (ii), $\tilde{u}(\cdot, j)$ is Lipschitz for every j. Since $\tilde{g}(\cdot, i)$ and $\tilde{h}(\cdot, i) + \sum\limits_{j\neq i}\lambda_{ij}\tilde{u}(\cdot, j)$ are Lipschitz, w is Lipschitz by Theorem 2.4 (i) if $r_i + \lambda_i$ is sufficient large.

However, $\tilde{X}$ is just a geometric Brownian motion, which is a special case of Regime-Switching model with $n = 1$. By Theorem 2.4 and Remark 2.5, if $r_i+\lambda_i > 0$, w is a Lipschitz continuous function and the unique viscosity solution with linear growth condition to the variational inequality

$$\min(-\hat{L}\hat{f}(x) - \tilde{h}(x, i) - \sum_{j\neq i}\lambda_{ij}\tilde{u}(x, j) + r_i\hat{f}(x),\ \hat{f}(x) - \tilde{g}(x, i)) = 0.$$

$\tilde{u}(\cdot, i)$ is a viscosity solution to this equation. By uniqueness, $w(\cdot)$ must coincide with $\tilde{u}(\cdot, i)$. Then, by Proposition 2.6, $\tilde{u}(\cdot, i)$ must be C^2 in $\tilde{C}_i$. Also $\tilde{u}(x, i)$ is C^1 in x at $\partial\tilde{C}_i$ if $\tilde{g}$ is C^1 at $\partial\tilde{C}_i$. The same must hold true for u, C and ∂C_i because of the transformational relationship between u, g, C, ∂C_i and $\tilde{u}$, $\tilde{g}$, $\tilde{C}$, $\partial\tilde{C}_i$. □

Remark 2.8. (i) In the proof of Proposition 2.7, we reduced the problem of proving regularity property for a Regime-Switching problem to proving regularity property for a one dimensional problems. The key is to treat the value function $\tilde{u}(x, j)$ for $j \neq i$ as a known function. We absorb the $\sum_{j\neq i}\lambda_{ij}\tilde{u}(x, j)$ term into the running pay-off and absorb $\lambda_i\tilde{u}(s, i)$ into the discount term. The idea of using a priori estimate of an unknown function resulted from a jump is used in a different way in Chapter 4 when we study the BNS model.

(ii) This method can be extended to more general SDE's where

$$\mathrm{d}X_t = \sigma(X_t, I_t)\mathrm{d}W_t + b(X_t, I_t)\mathrm{d}t,$$

provided σ and b satisfies the appropriate linear growth conditions. The restriction on r_i would be different depending on the functions σ and b.

We now return to the American put problem where

$$g(s,i) = (K-s)^+, \quad \tilde{g}(x,i) = (K-e^x)^+, \quad h(s,i) = \tilde{h}(x,i) = 0.$$

Since $g(s,i)$ does not depend on i, we use $g(s)$ to mean $g(s,i)$ without ambiguity. We now prove that the stopping region can be charactered by stopping levels b_i.

Lemma 2.9. *Recall the definition of $u(s,i)$ given by (2.3) on page 17 with*

$$g(s,i) = (K-s)^+ \quad \text{and} \quad h(s,i) = 0 \quad \text{for } i \in \{1,\dots,n\}, \tag{2.12}$$

and the definition of D_i given by (2.6) on page 18. Hence, $u(s,i)$ is the value function of a perpetual American put option under the Regime-Switching model and D_i is the stopping region for regime i. Then, the following results hold:

(i) $u(\cdot,i)$ is convex for $i \in \{1,\dots,n\}$,

(ii) there exists $b_i > 0$ such that D_i has the representation

$$D_i = \{s : s \le b_i\}.$$

Proof. First, we show convexity of $u(\cdot,i)$. For $\lambda \in (0,1)$, we have that

$$\begin{aligned}
u(\lambda s + (1-\lambda)s', i) &= \sup_\tau \mathbb{E} e^{-r\tau} g(S_\tau^{\lambda s+(1-\lambda)s',i}) \\
&= \sup_\tau \mathbb{E} e^{-r\tau} g((\lambda s + (1-\lambda)s') S_\tau^{1,i}) \\
&\le \sup_\tau \mathbb{E} e^{-r\tau} \lambda g(s S_\tau^{1,i}) + (1-\lambda) g(s' S_\tau^{1,i}) \\
&\le \sup_\tau \mathbb{E} e^{-r\tau} \lambda g(s S_\tau^{1,i}) + \sup_\tau \mathbb{E} e^{-r\tau} (1-\lambda) g(s' S_\tau^{1,i}) \\
&= \lambda \sup_\tau \mathbb{E} e^{-r\tau} g(S_\tau^{s,i}) + (1-\lambda) \sup_\tau \mathbb{E} e^{-r\tau} g(S_\tau^{s',i}) \\
&= \lambda u(s,i) + (1-\lambda) u(s',i),
\end{aligned}$$

where the first inequality follows by convexity of $g(\cdot)$. In the first and the penultimate line of the derivation, we use the fact that $S_\tau^{s,i} = s S_\tau^{1,i}$, which is a trivial consequence of (2.2).

We now show that the set D_i is non-empty. If the state i is an absorbing state, i.e. $\lambda_i = 0$, then $u(s,i)$ is the value of a perpetual American put problem under the Black-Scholes model. In that case, b_i has an explicit formula and is known to be non-zero. See, for example, [57, Section 25] for more details.

Suppose that the state i is not absorbing and D_i is empty. Starting from (s, i), it is not optimal to stop until the process I leaves the current state i. Define the stopping time T to be the first time I_t^i leaves the state i, i.e.

$$T = \inf\{t : I_t^i \neq i\}.$$

Let τ be the first hitting time of the stopping region, i.e.

$$\tau = \inf\{t : u(S_t^{s,i}, I_t^i) = g(S_t^{s,i})\}.$$

τ is a Markov time and the value function is attained at this Markov time by Remark 1.5(iii). Clearly, $T \leq \tau$. By the martingale property of $\{u(S_{t\wedge\tau}^{s,i}, I_{t\wedge\tau}^i) : 0 \leq t \leq \tau\}$, we have that

$$u(s,i) = \lim_{t\to\infty} \mathbb{E}e^{-r(t\wedge T)}u(S_{t\wedge T}^{s,i}, I_{t\wedge T}^i) = \mathbb{E}\lim_{t\to\infty} e^{-r(t\wedge T)}u(S_{t\wedge T}^{s,i}, I_{t\wedge T}^i) = \mathbb{E}e^{-rT}u(S_T^{s,i}, I_T^i),$$

where the limit is exchanged by the dominated convergence theorem since $|u|$ is bounded by K. It follows that

$$u(s,i) = \mathbb{E}e^{-rT}u(S_T^{s,i}, I_T^i) \leq K\mathbb{E}e^{-rT} = \frac{\lambda_i K}{r + \lambda_i},$$

where $\mathbb{E}e^{-rT} = \frac{\lambda_i}{r+\lambda_i}$ because T is exponentially distributed. However, by setting $s = K - \frac{\lambda_i K}{2(r+\lambda_i)}$, we have that

$$g\big(K - \tfrac{\lambda_i K}{2(r+\lambda_i)}\big) > \tfrac{\lambda_i K}{\lambda_i + r} \geq u\big(K - \tfrac{\lambda_i K}{2(r+\lambda_i)}, i\big),$$

which is a contradiction. We now define b_i by

$$b_i = \sup\{s : g(s) = u(s,i)\}.$$

Clearly $b_i < K$ as it is never optimal to exercise the option when the pay-off is zero. Now, consider a sequence of s_m in D_i such that $s_m \uparrow b_i$ as $m \to \infty$. By the continuity of $u(\cdot, i)$ and $g(\cdot)$, we must have $u(b_i, i) = g(b_i)$. Moreover, by Proposition 2.7, we must have $u(\cdot, i)$ is C^1 at b_i, i.e.,

$$\partial_1 u(b_i, i) = g'(b_i) = -1.$$

We now show for $s < b_i$, $g(s) = u(s,i)$. If there exists $s_1 < b_i$ such that

$$u(s_1, i) - g(s_1) = \epsilon > 0,$$

then it must be the case $s_1 \geq \epsilon$ since $u(s_1, i) < K$. Since u is convex, left and right derivatives must exist. Let $\partial_1 u(s_1-,i)$ denote the left derivative of u at s_1. By convexity of $u(\cdot,i)$, we must have

$$\begin{aligned} u(\epsilon/2,i) &\geq u(s_1,i) + \underbrace{\partial_1 u(s_1-,i)}_{\leq \partial_1 u(b_i,1)=-1}(\epsilon/2 - s_1) \\ &\geq K - s_1 + \epsilon - \epsilon/2 + s_1 = K + \epsilon/2, \end{aligned}$$

which is a contradiction. This proves $D_i = (0, b_i)$. □

The following corollary follows immediately from Proposition 2.7 and Lemma 2.9.

Corollary 2.10. *Under the same assumption as Lemma 2.9, the value function of the American put option $u(s,i)$ is the unique bounded solution to the free-boundary problem*

$$\begin{aligned} f(s,i) &= K - s && \textit{for } s \leq b_i \\ (L - r_i)f(s,i) &\leq 0 && \textit{for } s \leq b_i \\ (L - r_i)f(s,i) &= 0 && \textit{for } s > b_i \\ f(s,i) &> (K-s)^+ && \textit{for } s > b_i \\ f(b_i,i) &= K - b_i \\ \partial_1 f(b_i,i) &= -1 \end{aligned}$$

Remark 2.11. (i) The proof of Lemma 2.9 does not require the smooth pasting condition. The lemma can be extended to finite horizon. If we define $u(s,i,T)$ as

$$u(s,i,T) = \sup_{0 \leq \tau \leq T} \mathbb{E}\left[e^{-\int_0^\tau r(I_t^i)\mathrm{d}t}(K - S_\tau^{s,i})^+\right] \quad \text{for } s > 0,\ i \in \{1,\ldots,n\},$$

the region D_i as

$$D_i(T) = \{s : u(s,i,T) = g(s)\},$$

and the boundary $b_i(T)$ as

$$b_i(T) = \sup\{s : u(s,i,T) = g(s)\}.$$

It is possible to prove that $D_i(T)$ can be characterised by

$$D_i(T) = \{s : s \leq b_i(T)\}.$$

by a similar argument to the one presented in Lemma 2.9.

(ii) Corollary 2.10 has more conditions than equations (2.7) and (2.8). We are not able to show equations (2.7) and (2.8) have a unique solution without the extra conditions given in Corollary 2.10. Refer to Remark 2.3 and the discussions preceding it for more details.

(iii) In the case $\mu_i \leq r_i$, it is only necessary to check that a candidate solution $(\hat{b}_i, \hat{u}(s,i))$ satisfies $\hat{u}(s,i) \geq g(s)$ for $s > \hat{b}_i$. If $u(s,i) \geq g(s)$ for $s > \hat{b}_i$ holds, then $(L - r_i)\hat{u}(s,i) \leq 0$ holds automatically for $s < \hat{b}_i$. We refer to the argument in step (iii) of [39, Prop. 2]. This covers the risk-neutral case.

(iv) If $(\hat{b}_i, \hat{u}(s,i))$ satisfies the equations (2.7) and (2.8), then it corresponds to a stopping rule. In which case, we must have $u(s,i) \geq \hat{u}(s,i)$. We define τ by

$$\tau = \inf\{t : I_t = j,\ S_t \leq \hat{b}_j \text{ for any } j\}. \tag{2.13}$$

Since $\hat{u}(S_t^{s,i}, I_t^i)$ is in the domain of the generator for $t \leq \tau$, by Itô's formula, we have that

$$\hat{u}(s,i) = \lim_{t \to \infty} \mathbb{E} e^{-\int_0^{\tau \wedge t} r(I_q^i) \mathrm{d}q} \hat{u}(S_{\tau \wedge t}^{s,i}, I_{\tau \wedge t}^i).$$

Since $\hat{u}(S_\tau^{s,i}, I_\tau^i) = (K - S_\tau^{s,i})^+$ and $\hat{u}$ is bounded, by dominated convergence , we have that

$$\hat{u}(s,i) = \mathbb{E} e^{-\int_0^\tau r(I_q^i) \mathrm{d}q} (K - S_\tau^{s,i})^+.$$

This means $\hat{u}(s,i)$ is the expected reward of the following stopping rule: stop when the S is below $\hat{b}_i$ if I is currently in state i. $u(s,i)$ is the value function of the optimal stopping problem, so

$$u(s,i) \geq \hat{u}(s,i)$$

follows by definition.

(v) The necessity of the conditions in Corollary 2.10 and point (iii) above imply that we should restrict our attention to finding $(\hat{b}_i, \hat{u}(s,i))$ satisfying equations (2.7) and (2.8). If <u>all</u> solutions of (2.7) and (2.8) can be found (or indeed, if the solution can be shown to be unique), then there must be one solution $u^*(s,i)$ which dominates all other solutions. We must have that $u^* = u$. In this case, we would <u>not</u> need to verify the conditions

$$\begin{aligned} (L - r_i) u^*(s,i) &\leq 0 && \text{for } s \leq b_i, \\ u^*(s,i) &> (K - s)^+ && \text{for } s > b_i. \end{aligned}$$

They should hold automatically by the necessity of the conditions in Corollary 2.10.

2.3 Monotonicity of the value function under Regime-Switching and related models

A common feature of many stochastic volatility models is that the option price is increasing in the volatility parameter when everything else remains the same. The authors of [2] verified this for a number of models including the Regime-Switching model. Our goal is to extend this result for stochastic volatility models driven by continuous time Markov chains. This class of models contains, but is not restricted to the Regime-Switching models.

In this section, we shall proceed as follows. Firstly, we give a brief summary of the relevant assumptions, methods and results in [2]. Secondly, we discuss how their results for continuous time Markov chain driven models can be improved by weakening one of the assumptions. Thirdly, we restrict our attention to the Regime-Switching model. We prove a new monotonicity result and compare this with [2]. Every monotonicity result in Section 2.3.1 - 2.3.3 holds true for both finite and infinite horizon problems. In Section 2.3.4, we prove an extension of monotonicity results in Section 2.3.1 - 2.3.3 for infinite horizon problems.

2.3.1 Existing monotonicity results

We now introduce a probabilistic set-up given in [2]. Let $(S, Y) = (S_t, Y_t, t \geq 0)$ be a strong Markov process on a family of probability spaces $(\Omega, \mathcal{F}, \mathbb{P}_{s,y}, (s, y) \in \mathbb{R} \times \mathcal{S})$, which satisfies the SDE

$$\mathrm{d}S_t = a(S_t)Y_t\mathrm{d}W_t, \tag{2.14}$$

where W_t is a standard Brownian motion, and $a : \mathbb{R} \to \mathbb{R}$ is a measurable function. Consider the value function

$$v(s, y) = \sup_{0 \leq \tau \leq T} \mathbb{E}_{s,y}[e^{-r\tau}g(S_\tau)] \quad (s, y) \in \mathbb{R} \times \mathcal{S}, \tag{2.15}$$

where $\underline{\tau \text{ is a stopping time at which } g(S_\tau) \geq 0}$ and $T \in [0, \infty]$. For $T < \infty$, this set of stopping is guaranteed to be non-empty when $g \geq 0$. We assume that g is chosen such that $v(s, y)$ is well defined. We are interested in the case where Y is a finite state continuous time Markov chain and $\mathcal{S} = \{y_1, \ldots, y_n\}$. We denote the Q-matrix of Y by Q and its entries by q_{ij}. For $i \neq j$, we use q_{ij} to denote the jump rates from i to j. For $i = j$, q_{ii} is the total rate leaving state i multiplied by -1. In addition, we assume that $0 < y_1 < y_2 < \cdots < y_n$.

Remark 2.12. Recall the Regime-Switching SDE (2.1). If we set $r(i) = r$, $\mu(i) = 0$, $\sigma(i) = y_i$ for $i \in \{1, \ldots, n\}$, then this coincides with (2.14) with $a(s) = s$. The process Y and I are related via $Y = \sigma(I)$. The Q-matrix Λ in (2.1) corresponds to the matrix Q in the set-up of this section with $q_{ij} = \lambda_{ij}$. The optimal stopping problem associated with the

Regime-Switching model (2.3) coincides with (2.15) when $g(s,i)$ in (2.3) is independent of i and $h(s,i)=0$. In this case $v(s,\sigma(i))=u(s,i)$.

In some sense, the model (2.15) is more general than (2.1) as $a(\cdot)$ is ato be a general function. On the other hand, (2.15) does not allow a regime dependent drift term, because the coupling method relies on the relationship between continuous martingales and Brownian motion.

The method in [2] considers the martingale $M_t = \int_0^t Y_u \mathrm{d}W_u$. Time-changing by the inverse of $\langle M \rangle$ yields

$$G_t = s + \int_0^t a(G_u)\mathrm{d}B_u,$$

where $B = M \circ \langle M \rangle$ is an $\mathcal{F}_{\langle M \rangle_t^{-1}}$ Brownian motion. The process $Y \circ \langle M \rangle_t^{-1}$ is a continuous time Markov chain independent of B (living on the state space $\mathcal{S}$), with transition rates

$$\tilde{q}_{ij} = y_i^{-2} q_{ij}.$$

The authors of [2] then constructed a coupled process (Z, Z') on a probability space such that Z and Z' are continuous time Markov chains with transition rates $\tilde{q}_{ij}$ satisfying the condition

$$y = Z_0 \leq Z_0' = y' \quad \Rightarrow \quad Z_t \leq Z_t' \ \text{ for } \ t \geq 0. \tag{2.16}$$

This coupling is available when the Q-matrix of the time-changed Markov chain is skip-free. This means

$$\tilde{q}_{ij} = 0 \ \text{ for } \ j \neq i-1, \ i, \ i+1,$$

which is equivalent to

$$q_{ij} = 0 \ \text{ for } \ j \neq i-1, \ i, \ i+1.$$

By considering an optimal stopping problem in term of G, Z, Z', B, the following theorem can be proved.

Theorem 2.13 (Theorem 2.5 of [2]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable gain function such that $\{g \geq 0\} \neq \emptyset$. Recall the definition of v given by (2.15). Suppose Y is skip-free, then*

$$v(s,y) \leq v(s,y') \quad \text{for all } y, y' \in \mathcal{S} \text{ such that } y \leq y'.$$

2.3.2 Application of order preserving coupling in monotonicity of value function

The proof of Theorem 2.13 requires the existence of a coupling, under which the time changed Markov chain Z satisfies (2.16). The skip-free condition is an obvious sufficient condition for such coupling to exist, but it is rather restrictive. Under the skip-free condition, we can take the coupling which let Z and Z' jump independently until the first time they coalesce. An obvious way to extend the current result is to find an alternative sufficient condition for constructing an order preserving coupling. We now state a sufficient condition for (2.16).

Proposition 2.14. *Let Q be a Q-matrix for a Markov chain on state space $\mathcal{S} = \{y_1, \dots, y_n\}$ such that $0 < y_1 < \dots < y_n$ and q_{ij} denote its entries. If the q_{ij} satisfies the following two conditions:*

(c1) for $1 \le i < i' < j \le n$, $\sum_{k=j}^{n} q_{ik} \le \sum_{k=j}^{n} q_{i'k}$,

(c2) for $1 \le j < i < i' \le n$, $\sum_{k=1}^{j} q_{ik} \ge \sum_{k=1}^{j} q_{i'k}$,

then there exists a coupling under which $Y'_t \ge Y_t$ if $Y'_0 \ge Y_0$.

This turns out to be a well-known result about Markov chains. See, for example [48], for a proof under a more general setting. We give an explicit construction for such a coupling on page 47 of the Chapter Appendix.

The conditions (c1) and (c2) mean, for any pair of states i and i' such that $i < i'$,

(i) for all $j' > i'$, the total jump rate from y_i to $\{y_{j'}, y_{j'+1} \dots, y_n\}$ is less than the total jump rate from $y_{i'}$ to the same set of states.;

(ii) for all $j < i$, the total jump rate from y_i to $\{y_1, y_2 \dots, y_j\}$ is greater than the total jump rate from $y_{i'}$ to the same set of states.

The idea behind the coupling we construct on page 47 is as follows: if the process (Y, Y') is currently in state $(y_i, y_{i'})$ such that $i < i'$, then any jumps by Y to states $\{y_{i+1}, \dots, y_{i'}\}$ and any jump by Y' to $\{y_i, \dots, y_{i-1}\}$ occurs independently. A jump by Y from state i to a state $j > i'$ must occur at the same time as a jump by Y' to a state $j' \ge j$. Similarly all jumps by Y' to a state $y_{j'} < y_i$ occurs at the same time as a jump by Y to some $y_j < y_{j'}$.

Remark 2.15. We now give two examples where (c1) and (c2) on page 30 are satisfied.

(i) Any Q-matrix satisfying the skip-free condition automatically satisfy (c1) and (c2). This include the zero-matrix when there is no regime change and any two-state Q-matrix.

(ii) Any Q-matrix satisfying the conditions

  (c1') for all $1 \le i < i' < j \le n$, $q_{ij} \le q_{i'j}$,

  (c2') for all $1 \le j < i < i' \le n$, $q_{ij} \ge q_{i'j}$, automatically satisfies (c1) and (c2). An example of such matrix is

$$\begin{pmatrix} -2 & 1 & 1 \\ 3 & -5 & 2 \\ 2 & 2 & -4 \end{pmatrix}$$

(c1') and (c2') are sufficient conditions for (c1) and (c2) respectively. They are equivalent to (c1) and (c2) if the number of states is less or equal to 3.

By the argument preceding Theorem 2.13, we have the following corollary of Proposition 2.14.

Corollary 2.16. *Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable gain function such that $\{g \geq 0\} \neq \emptyset$. Let Y be a Markov chain living on the state space $\mathcal{S} = \{y_1, \ldots, y_n\}$ with transition rates q_{ij}. If the time changed Markov chain with jump rates*

$$\tilde{q}_{ij} = y_i^{-2} q_{ij},$$

satisfies (c1) and (c2) on page 30, explicitly, that is,

(c1') for $1 \leq i < i' < j \leq n$, $\sum_{k=j}^{n} y_i^{-2} q_{ik} \leq \sum_{k=j}^{n} y_{i'}^{-2} q_{i'k}$,

(c2') for $1 \leq j < i < i' \leq n$, $\sum_{k=1}^{j} y_i^{-2} q_{ik} \geq \sum_{k=1}^{j} y_{i'}^{-2} q_{i'k}$,

then the value function v defined by (2.15) on page 28 satisfies

$$v(s, y) \leq v(s, y') \quad \text{for all } y, y' \in \mathcal{S} \text{ such that } y \leq y'.$$

Remark 2.17. A related problem to (2.15) with financial application is

$$v(s, y) = \sup_{0 \leq \tau \leq T} \mathbb{E}_{s,y}[e^{-r\tau} g(e^{\mu t} S_\tau)] \quad (s, y) \in \mathbb{R} \times \mathcal{S}.$$

By argument found in Corollary 5.1 of [2], Corollary 2.16 holds for $v(s, y)$ subject to the extra constraints that $\mu \geq 0$ (or $\leq$) if g is a decreasing (or increasing) positive function. In the risk-neutral case, we have $r = \mu$.

We now prove a new monotonicity result for the Regime-Switching model. Recall the Regime-Switching process S_t driven by the stochastic differential equation

$$\mathrm{d}S_t = \sigma(I_t) S_t \mathrm{d}W_t + \mu(I_t) S_t \mathrm{d}t,$$

where $\sigma : \{1, \ldots, n\} \to (0, \infty)$ and $\mu : \{1, \ldots, n\} \to (-\infty, \infty)$ are known functions.

Proposition 2.18. *Recall the set-up of the Regime Switching model. Let W be a Brownian motion and I be an n state continuous time Markov chain taking values in $\{1, \dots, n\}$. Let S_t be driven by the stochastic differential equation*

$$dS_t = \sigma(I_t) S_t dW_t + \mu(I_t) S_t dt,$$

where $\sigma : \{1, \dots, n\} \to (0, \infty)$ and $\mu : \{1, \dots, n\} \to (-\infty, \infty)$.

Let g be a function such that $g : (0, \infty) \times \{1, \dots, n\} \to \mathbb{R}$. Consider the value function

$$u(s, i, T) = \sup_{0 \le \tau \le T} \mathbb{E}\left[e^{-\int_0^\tau r(I_t^i) dt} g(S_\tau^{s,i}, I_\tau^i) \right] \quad \text{for } s > 0,\ i \in \{1, \dots, n\},$$

where $r : \{1, \dots, n\} \to (0, \infty)$ and the supremum is taken over the class of all stopping times bounded above by T, where $T \in (0, \infty)$. In addition, we assume:

(a1) $\sigma_1 \le \sigma_2 < \cdots \le \sigma_n$,

(a2) $g(\cdot, 1) \le g(\cdot, 2) \le \cdots \le g(\cdot, n)$,

(a3) $g(\cdot, i)$ *is convex for all* $i \in \{1, \dots, n\}$,

(a4) one of the two following condition holds:

- $r_1 \ge r_2 \ge \cdots \ge r_n > 0$ *and* $g \ge 0$,
- $r_1 = r_2 = \cdots = r_n$,

(a5) one of the three following conditions holds:

- $\mu_1 = \mu_2 = \cdots = \mu_n$,
- *If $g(\cdot, i)$ is increasing for every i,* $\mu_1 \le \mu_2 \le \cdots \le \mu_n$,
- *If $g(\cdot, i)$ is decreasing for every i,* $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$,

(a6) The Q-matrix of I satisfies (c1) and (c2) on page 30.

The value function u satisfies

$$u(s, i, T) \le u(s, i', T) \quad \text{for all } i,\ i' \in \{1, \dots, n\} \text{ such that } i \le i'.$$

Remark 2.19. Proposition 2.18 holds for $T = \infty$ if the condition

$$\lim_{T \to \infty} u(s, i, T) = u(s, i, \infty)$$

This is certainly the case if g is bounded. We use the notation $u(s, i)$ as a shorthand for $u(s, i, \infty)$.

Proposition 2.18 is a trivial consequence of Lemma 2.21 and Proposition 2.22. Before we proceed to prove Proposition 2.22, we define the following function.

Definition 2.20. *Consider the probabilistic set-up of Proposition 2.18. We define* $U : (0,\infty) \times \{1,\ldots,n\} \times (0,\infty) \times \mathbb{N}^+ \to (0,\infty)$ *by*

$$U(s,i,T,m) = \sup_{\tau \in \{\frac{T}{m},\frac{2T}{m},\ldots,T\}} \mathbb{E}\left[e^{-\int_0^\tau r(I_t^i)dt} g(S_\tau^{s,i}, I_\tau^i)\right],$$

where τ *is a stopping time.*

U is the price of a Bermudan option with gain function g and allowed exercise times $\{\frac{T}{m},\frac{2T}{m},\ldots,T\}$. In order to prove Proposition 2.18, we first prove that $U(s,i,T,n)$ is monotone in i and then the monotonicity of $u(s,i,T)$ follows by the next lemma.

Lemma 2.21. *Under the set-up of Proposition 2.18, we have the following convergence result*

$$\lim_{l\to\infty} U(s,i,T,2^l) = u(s,i,T).$$

The proof of this lemma is fairly straightforward and is found in the Appendix section of this chapter on page 51.

Proposition 2.22. *Let* U *be the function defined in Definition 2.20. In addition, we assume that assumptions (a1) - (a6) of Proposition 2.18 holds. Then,* U *has the following properties*

(p1) $U(s,\cdot,T,m)$ *is increasing.*

(p2) $U(\cdot,i,T,m)$ *is convex.*

(p3) If $g(\cdot,i)$ *is increasing (or decreasing), then* $U(\cdot,i,T,m)$ *is increasing (or decreasing).*

Remark 2.23. Throughout the proof of Proposition 2.22, (p1) is the monotonicity result we need to prove for Proposition 2.18. To prove (p1), we need to prove that $u(s,i,T,m) \le u(s,i',T,m)$ whenever $i < i'$.

By assumption (a6) and Proposition 2.14, for every pair of (i,i') such that $i < i'$, we can construct a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, under which there is a pair $(\tilde I^i,\tilde I^{i'})$ such that $\tilde I_t^i \le \tilde I_t^{i'}$ for all $t \ge 0$. Moreover, the marginal distributions of I^i and $I^{i'}$ are the same as the marginal distributions of $\tilde I^i$ and $\tilde I^{i'}$, respectively. Furthermore, let $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ support a Brownian motion $\tilde W$ and define process $\tilde S$ by the SDE

$$\mathrm{d}\tilde S_t = \sigma(\tilde I_t)\tilde S_t \mathrm{d}\tilde W_t + \mu(\tilde I_t)\tilde S_t \mathrm{d}t.$$

Under this new set-up, we still have

$$u(s,i,T) = \sup_{0\le\tau\le T} \tilde{\mathbb{E}}\left[e^{-\int_0^\tau r(\tilde I_t^i)\mathrm{d}t} g(\tilde S_\tau^{s,i}, \tilde I_\tau^i)\right].$$

The same is true if we replace i, $\tilde{I}^i$ and $\tilde{S}^{s.i}$ by i', $\tilde{I}^{i'}$ and $\tilde{S}^{s.i'}$. In the proof of Proposition 2.22, we work with $(\tilde{I}^i, \tilde{I}^{i'})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, but we drop the tilde from our notation by an abuse of notation.

Proof of **Proposition 2.22**. The proof is by induction. We first prove convexity of $U(\cdot, i, T, 1)$. For convexity, the argument hinges on $S_t^{s,i} = sS_t^{1,i}$. See Lemma 2.9 for a similar argument. For $s, s' \in (0, \infty)$, $\lambda \in (0, 1)$, we have that

$$\begin{aligned} U(\lambda s + (1-\lambda)s', i, T, 1) =& \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(S_T^{\lambda s+(1-\lambda)s',i}, I_T^i)\right] \\ &\leq \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t}(\lambda g(S_T^{s,i}, I_T^i) + (1-\lambda) g(S_T^{s',i}, I_T^i))\right] \\ &= \lambda U(s, i, T, 1) + (1-\lambda) U(s', i, T, 1), \end{aligned}$$

where the inequality holds by convexity of $g(\cdot, i)$. We now prove $U(s, i, T, 1) \leq U(s, i', T, 1)$ for $i < i'$. We introduce the following notations: if Y has a log-normal distribution with $\log(Y) \sim N(-\frac{1}{2}y, y)$, we denote the distribution function of Y by N_y. For $i' > i$, we use $F_{i,i',T}(y_1, y_2)$ to denote the joint distribution function of Y_1 and Y_2, where

$$Y_1 = \int_0^T \sigma(I_t^{i'})^2 \mathrm{d}t \quad \text{and} \quad Y_2 = \int_0^T \sigma(I_t^i)^2 \mathrm{d}t.$$

First, we have

$$\begin{aligned} &\mathbb{E}[e^{-\int_0^T r(I_t^{i'})\mathrm{d}t} g(S_T^{s,i'}, I_T^{i'}) - e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(S_T^{s,i}, I_T^i)] \\ &\quad = \mathbb{E}[\mathbb{E}[e^{-\int_0^T r(I_t^{i'})\mathrm{d}t} g(S_T^{s,i'}, I_T^{i'}) - e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(S_T^{s,i}, I_T^i) | \sigma(I_t : 0 \leq t \leq T)]] \\ &\quad = \mathbb{E}[\mathbb{E}[e^{R_1(T)} g(se^{M_1(T)}\eta_1, \alpha_T) - e^{R_2(T)} \underbrace{g(se^{M_2(T)}\eta_2, \beta_T)}_{\leq g(se^{M_2(T)}\eta_2, \alpha_T) \text{ by (a2)}}]|_{\alpha_t = I_t^{i'}, \beta_t = I_t^i}] \\ &\quad \geq \mathbb{E}[\mathbb{E}[e^{R_1(T)} g(se^{M_1(T)}\eta_1, \alpha_T) - e^{R_2(T)} g(se^{M_2(T)}\eta_2, \alpha_T)]|_{\alpha_t = I_t^{i'}, \beta_t = I_t^i}], \end{aligned} \tag{2.17}$$

where η_1 and η_2 have log-normal distribution with distribution functions N_{y_1} and N_{y_2}, and

$$y_2 = \int_0^T \sigma(\beta_t)^2 \mathrm{d}t, \quad y_1 = \int_0^T \sigma(\alpha_t)^2 \mathrm{d}t,$$

$$R_1(T) \stackrel{\text{def}}{=} -\int_0^T r(\alpha_t)\mathrm{d}t \geq -\int_0^T r(\beta_t)\mathrm{d}t \stackrel{\text{def}}{=} R_2(T)$$

$$M_1(T) \stackrel{\text{def}}{=} \int_0^T \mu(\alpha_t)\mathrm{d}t, \quad M_2(T) \stackrel{\text{def}}{=} \int_0^T \mu(\beta_t)\mathrm{d}t.$$

It follows from assumptions (a4), (a5) and (2.17) that

$$\begin{aligned}
&\mathbb{E}[e^{-\int_0^T r(I_t^{i'})\mathrm{d}t} g(S_T^{s,i'}, I_T^{i'}) - e^{-\int_0^T r(I_t^{i})\mathrm{d}t} g(S_T^{s,i}, I_T^{i})] \\
&\qquad \geq \mathbb{E}[\mathbb{E}[e^{R_1(T)} g(se^{M_1(T)}\eta_1, \alpha_T) - e^{R_1(T)} g(se^{M_2(T)}\eta_2, \alpha_T)]|_{\alpha_t = I_t^{i'}, \beta_t = I_t^i}] \\
&\qquad \geq \mathbb{E}[e^{R_1(T)} \mathbb{E}[g(se^{M_1(T)}\eta_1, \alpha_T) - g(se^{M_1(T)}\eta_2, \alpha_T)]|_{\alpha_t = I_t^{i'}, \beta_t = I_t^i}] \qquad (2.18)
\end{aligned}$$

This allows us to write (2.18) as the following integral:

$$\int_0^\infty e^{R_1(T)} \left[\int_0^\infty g(se^{M_1(T)}p, \alpha_T) N_{y_1}(\mathrm{d}p) - \int_0^\infty g(se^{M_1(T)}p, \alpha_T) N_{y_2}(\mathrm{d}p) \right] \mathrm{d}F_{i,i',T}(y_1, y_2).$$

Now consider an independent probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ supporting two independent random variables ζ_2 and η with distribution function N_{y_2} and $N_{(y_1 - y_2)}$. Let $\log(\zeta_1) = \log(\zeta_2) + \log(\eta)$. Note that $\log(\zeta_1) \sim N(-\frac{1}{2}y_1, y_1)$, so ζ_1 has density function N_{y_1}. We have, for any $s > 0, y_1 > y_2$,

$$\begin{aligned}
&\int_0^\infty g(se^{M_1(T)}p, \alpha_T) N_{y_1}(\mathrm{d}p) - \int_0^\infty g(se^{M_1(T)}p, , \alpha_T) N_{y_2}(\mathrm{d}p) \\
&\qquad = \hat{\mathbb{E}}[g(se^{M_1(T)}\zeta_1, \alpha_T) - g(se^{M_1(T)}\zeta_2, \alpha_T)] \\
&\qquad = \hat{\mathbb{E}}[g(se^{M_1(T)}\zeta_2\eta, \alpha_T) - g(se^{M_1(T)}\zeta_2, \alpha_T)] \\
&\qquad = \hat{\mathbb{E}}[\hat{\mathbb{E}}[g(se^{M_1(T)}\zeta_2\eta, \alpha_T) - g(se^{M_1(T)}\zeta_2, \alpha_T) | \sigma(\zeta_2)]] \\
&\qquad = \hat{\mathbb{E}}[\underbrace{\hat{\mathbb{E}}[g(se^{M_1(T)}u\eta, \alpha_T) - g(se^{M_1(T)}u, \alpha_T)]}_{\geq 0 \text{ by Jensen's inequality}} |_{u=\zeta_2}] \\
&\qquad \geq 0 \qquad (2.19)
\end{aligned}$$

It follows from (2.18) and (2.19) that

$$\mathbb{E}[e^{-\int_0^T r(I_t^{i'})\mathrm{d}t} g(S_T^{s,i'}, I_T^{i'}) - e^{-\int_0^T r(I_t^{i})\mathrm{d}t} g(S_T^{s,i}, I_T^{i})] \geq 0, \qquad (2.20)$$

If $g(\cdot, i)$ is increasing, then, for $s \leq s'$, we have that

$$\begin{aligned}
U(s, i, T, 1) &= \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(S_T^{s,i}, I_T^i)\right] = \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(sS_T^{1,i}, I_T^i)\right] \\
&\leq \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(s'S_T^{1,i}, I_T^i)\right] = \mathbb{E}\left[e^{-\int_0^T r(I_t^i)\mathrm{d}t} g(S_T^{s',i}, I_T^i)\right] = u(s', i, T, 1).
\end{aligned}$$

By a similar argument, if $g(\cdot, i)$ is decreasing, $U(s, i, T, 1)$ is decreasing. Now, we have shown $U(s, i, T, 1)$ has properties (p1) - (p3). Now suppose $U(s, i, T, m-1)$ have properties (p1)

- (p3). We define the function

$$w(s,i,T,m) = \max(g(s,i), U(s,i,\tfrac{T(m-1)}{m}, m-1)).$$

$w(s,i,T,m)$ has a number of properties

(p1') $w(s,\cdot,T,m)$ is increasing (by induction hypothesis).

(p2') $w(\cdot,i,T,m)$ is convex as it is the maximum of two convex functions.

(p3') $w(\cdot,i,T,m)$ is increasing (or decreasing) if $g(\cdot,i)$ is increasing (or decreasing).

(p4') $w \geq 0$ if $g \geq 0$.

By Bellman's principle, we have that

$$U(s,i,T,m) = \mathbb{E}\Big[e^{-\int_0^{T/m} r(I_t^i)\mathrm{d}t} w(S^{s,i}_{T/m}, I^i_{T/m}, T, m)\Big].$$

We now define a function $\tilde{U}$ in the same way as we defined U in Definition 2.20. The time horizon is T/m and the gain function is $w(\cdot,\cdot,T,m)$. Hence

$$\tilde{U}(s,i,1,T/m) = \mathbb{E}\Big[e^{-\int_0^{T/m} r(I_t^i)\mathrm{d}t} w(S^{s,i}_{T/m}, I^i_{T/m}, T, m)\Big].$$

Properties (p1') - (p4') of w mean assumptions (a1) - (a6) are satisfied when g is replaced by w. Hence, by applying the $m = 1$ case of the induction with $g(\cdot,\cdot)$ replaced by $w(\cdot,\cdot,T,m)$, $\tilde{U}(s,i,1,T/m)$ has property (p1) - (p3). Since $U(s,i,T,m) = \tilde{U}(s,i,1,T/m)$, $U(s,i,T,m)$ has properties (p1) - (p3) as required. □

We have now proven Lemma 2.21 and Proposition 2.22. This means Proposition 2.18 must hold.

2.3.3 Comparison of monotonicity results and application to American put

Recall the correspondence between the parameters in Corollary 2.16 and Proposition 2.18 given in Remark 2.12. In this section, we use the notation for the regime switching problem given Proposition 2.18. We summarise the conditions needed for Corollary 2.16 and Proposition 2.18 in Table 1 on page 37. The conditions in Table 1 are in addition to the common constraint $\sigma_1^2 \leq \cdots \leq \sigma_n^2$.

Corollary 2.16 and Proposition 2.18 are generally very different, even for Regime-Switching models. This is because the restrictions placed on the jump matrix are very different. The condition (c1) is a stronger assumption than (c1'), but (c2) is a weaker assumption than (c2'). They only seem to coincide when the jump matrix is skip-free.

Both Corollary 2.16 and Proposition 2.18 can be applied to the American put problem in finite or infinite horizon. We examine this important case in Example 2.25 below.

	Corollary 2.16	Proposition 2.18
SDE	$\mathrm{d}S_t = a(S_t)\sigma(I_t)\mathrm{d}W_t$	$\mathrm{d}S_t = \sigma(I_t)S_t\mathrm{d}W_t + \mu(I_t)S_t\mathrm{d}S_t$
Pay-off function	$e^{-rt}g(e^{\mu t}S_t^{s,i})$ $\{s : g(s) > 0\} \neq \emptyset$ $g \geq 0$ in finite horizon	$e^{-\int_0^t r(I_u^i)\mathrm{d}u}g(S_t^{s,i}, I_t^i)$ $g(s,\cdot)$ is increasing. $g(\cdot,i)$ is convex
Restriction on r or r_i	$r > 0$	$r_1 = \cdots = r_n > 0$ in general. $r_1 \geq \cdots \geq r_n > 0$ and $g \geq 0$ $0 < r_1 \leq \cdots \leq r_n$ and $g \leq 0$
Restriction on μ or μ_i	$\mu = 0$ in general $\mu \geq 0$ if g is decreasing $\mu \leq 0$ if g is increasing	$\mu_1 = \cdots = \mu_n$ in general $\mu_1 \geq \cdots \geq \mu_n$ if g is decreasing $\mu_1 \leq \cdots \leq \mu_n$ if g is increasing
Jump matrix restriction	Condition (c1') and (c2')	Condition (c1) and (c2)

Table 1: Comparison between Corollary 2.16 and Proposition 2.18

Remark 2.24. The monotonicity property of the value function not only offers us better understanding about the value function, but also has important practical implications for the numerical schemes used to estimate the value of the options.

Firstly, monotonicity results can reduce the complexity of numerical schemes. This point was noted in [2], where the authors also discussed the perpetual American put problem under the Regime Switching model. In infinite horizon, recall that the value function satisfies the free-boundary problem given in Remark 2.10. The numerical scheme proposed [39] tries to find stopping thresholds b_i without assumptions on the ordering. By examining every possible arrangement, the problem has exponential complexity. However, $u(s,i) \geq u(s,i')$ implies $b_i < b_{i'}$. Hence, the problem has linear complexity in the number of states when the monotonicity property is known.

Secondly, monotonicity results can be used to determine the validity of numerical schemes. In [12], Buffington and Elliott analysed the American put problem for two-state Regime Switching model in finite horizon. As discussed in Remark 2.11 (i), the stopping regions for finite horizon American put are characterised by stopping boundaries $b_1(t)$ and $b_2(t)$. For a fixed time horizon T, the authors of [12] performed analysis on the value function based on the assumption that either $b_1(t) \geq b_2(t)$ or $b_2(t) \geq b_1(t)$ for all $t \in [0,T]$. They also proposed a numerical scheme assuming this holds. This assumption is not a trivial one and it is unclear to us whether this always holds. Like in the perpetual case, monotonicity of the value function implies the monotonicity of stopping boundary. Hence, we know the algorithm in [12] can be used safely for some choices of parameters.

Example 2.25. Let $u(s,i,T)$ be the value function of the American put option under Regime-Switching, i.e.,

$$u(s,i,T) = \sup_{0\le\tau\le T} \mathbb{E}e^{-\int_0^\tau r(I_t^i)\mathrm{d}t}(K - S_\tau^{s,i})^+.$$

We want to know what are some sufficient conditions for $u(\cdot,1) \le u(\cdot,2) \le \cdots \le u(\cdot,n)$. Applying the result of Proposition 2.18 with $g(s,i) = (K-s)^+$, we have

(i) $\sigma_1 \le \cdots \le \sigma_n$,

(ii) $r_1 \ge \cdots \ge r_n > 0$,

(iii) $\mu_1 \ge \cdots \ge \mu_n$.

(iv) The Q-matrix satisfies the coupling condition (c1) and (c2).

This result is consistent with one of the numerical examples in [15] and [42]. In their example, $n = 4$, $T = 1$, the parameters r_i's and μ_i's satisfy conditions (ii) and (iii), and the Q-matrix is given by

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

Applying Corollary 2.16 to gain function $g(s) = (K-s)^+$, we have

(i') $\sigma_1 \le \cdots \le \sigma_n$,

(ii') $r_1 = \cdots = r_n > 0$,

(iii') $\mu_1 = \cdots = \mu_n > 0$,

(iv') The time changed Q-matrix satisfies the coupling condition (c1) and (c2).

Condition (iii') is somewhat restrictive, but Corollary 2.16 can be used to improve condition (iii') by a simple transformation. Let S_t be of the form

$$S_t^{s,i} = s\exp\left(\int_0^t \sigma(I_u)\mathrm{d}W_u + c\int_0^t \sigma(I_t)^2\mathrm{d}t\right),$$

where $c \neq 0$. In this case $\mu_i = c\sigma_i^2$ for a non-zero constant c. We define α such that $(S_t)^\alpha$ is a martingale. Since

$$(S_t^{s,i})^\alpha = s^\alpha \exp\left(\alpha \int_0^t \sigma(I_u)\mathrm{d}W_u + \alpha c \int_0^t \sigma(I_t)^2 \mathrm{d}t\right),$$

we must have

$$-\frac{1}{2}\alpha^2 = \alpha c \quad \Rightarrow \quad \alpha = -2c.$$

Hence, if we let $\tilde{S}_t^{s^{-2c},i} = (S_t^{s,i})^{-2c}$, then $\mathrm{d}\tilde{S}_t = -2c\sigma(I_t)\tilde{S}_t\mathrm{d}W_t$. Moreover, we can write

$$(K - e^{\mu_0 t}S_t^{s,i})^+ = (K - ((e^{\mu_0 t}S_t^{s,i})^{-2c})^{-\frac{1}{2c}})^+ = (K - (e^{-2c\mu_0 t}(\tilde{S}_t^{s^{-2c},i}))^{-\frac{1}{2c}})^+.$$

Since the transformation from S to $\tilde{S}$ is bijective, we have that

$$u(s,i,T) = \sup_{0\leq\tau\leq T} \mathbb{E}e^{-r\tau}(K - e^{\mu_0 t}S_\tau^{s,i})^+ = \sup_{0\leq\tau\leq T} \mathbb{E}e^{-r\tau}(K - (e^{-2c\mu_0 t}\tilde{S}_\tau^{s^{-2c},i})^{-\frac{1}{2c}})^+,$$

where the supremum is taken over all stopping times with respect to the same filtration.

Let $\mu_0 \geq 0$ and $c < 0$, then $g(\cdot) = (K - (\cdot)^{-\frac{1}{2c}})^+$ is decreasing and $-2c\mu_0 \geq 0$. We can apply Corollary 2.16 to the gain function of the form appearing in Remark 2.17 (i) on page 31 with $\tilde{S}$ replacing S, $a(\tilde{S}) = -2c\tilde{S}$, $\mu = -2c\mu_0$ and $g(\cdot) = (K - (\cdot)^{-\frac{1}{2c}})^+$. We can therefore replace (iv') with an improved condition

$$\mu_i = c\sigma_i^2 + \mu_0,$$

where $c < 0$ and $\mu_0 \geq 0$. For $c > 0$, $\mu_0 \geq 0$, we have $-2c\mu_0 \leq 0$ instead of $-2c\mu_0 \geq 0$, but $g(\cdot) = (K - (\cdot)^{-\frac{1}{2c}})^+$ is now increasing, so the result still holds. For $c = 0$, $\mu_0 \geq 0$, we can just consider the martingale

$$\mathrm{d}\tilde{S}_t = \sigma(I_t)\mathrm{d}W_t$$

and $g(\cdot) = (K - \exp(\cdot))^+$. Hence, we can replace (iv') with the condition

$$\mu_i = c\sigma_i^2 + \mu_0, \quad c \in (-\infty, \infty),\ \mu_0 \geq 0. \tag{2.21}$$

For a two-state model, any valid Q-matrix is skip-free. By rearranging condition (2.21), we get

$$\mu_2 = \frac{\sigma_2^2}{\sigma_2^1}\mu_1 + \left(1 - \frac{\sigma^2}{\sigma_1^2}\right)\mu_0 \quad \text{for } \mu_0 \geq 0.$$

This means

$$\mu_2 \leq \frac{\sigma_2^2}{\sigma_1^2}\mu_1.$$

We can compare this with $\mu_2 \leq \mu_1$ from Proposition 2.18. For $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $r_1 = r_2 > 0$, the values of μ_1 and μ_2 for which $u(s,1) \leq u(s,2)$ are illustrated by the shaded regions in Figure 1. The region shaded in blue is deduced from inequality (2.21) and the region shaded in red is deduced from Proposition 2.18.

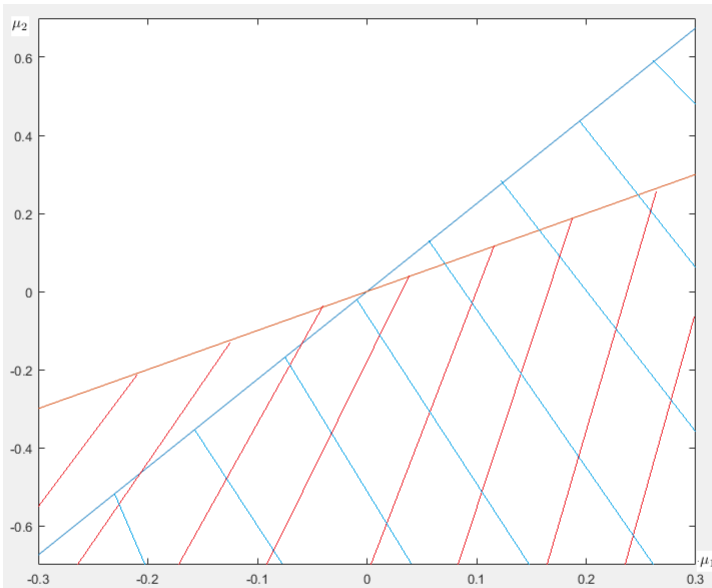


Figure 1: Values of μ_1 and μ_2 for which $u(s,2) \geq u(s,1)$

Remark 2.26. (i) (2.21) may seem counter-intuitive at first. For example, let $n = 2$, for large positive values of c, we have that $\mu_2 \gg \mu_1$. One would expect that if the positive drift is much larger for $i = 2$ than for $i = 1$, this may erase the higher option value gained from $\sigma_2 > \sigma_1$.

However, it turns out (2.21) is consistent with the solution of the American put problem under Black-Scholes. Since zero-matrix satisfies (c1) and (c2), all result we have proven here must hold when the jump matrix is the zero-matrix. In this case, the solution of the Regime-Switching problem with $T = \infty$ coincide with the solution of two disjoint McKean problems. The solution is given by

$$u(s,i) = \begin{cases} K - s & : \quad s \in (0, b_i], \\ (K - b_i)\left(\dfrac{s}{b_i}\right)^{\gamma_i} & : \quad s \in (b_i, \infty), \end{cases}$$

where $b_i = -\gamma_i K/(1-\gamma_i)$, and γ_i stands for the negative root of the quadratic equation

$$\frac{1}{2}\sigma_i^2\gamma^2 + (\mu_0 + (c - \tfrac{1}{2})\sigma_i^2)\gamma - r = 0.$$

Explicitly,

$$\gamma_i = -\sqrt{(c - \tfrac{1}{2} + \tfrac{\mu_0}{\sigma_i^2})^2 + \tfrac{2r}{\sigma_i^2}} - (c - \tfrac{1}{2} + \tfrac{\mu_0}{\sigma_i^2}). \tag{2.22}$$

It can be checked that $|\gamma_2| < |\gamma_1|$, which implies $u(s,2) \geq u(s,1)$.

(ii) Fix σ_1 and σ_2, for every choice of r and generator matrix Q, there is set of parameters μ_1 and μ_2 such that $u(s,1) \leq u(s,2)$. Let us denote this set by $M(r,Q)$. It is clear from expression (2.22) that $M(r,0)$ is a strict superset of the shaded region in Figure 1. However, it is unclear whether the shaded region is $\cap_{r,Q} M(r,Q)$, where the intersection is taken over all positive values of r and valid two-state Q-matrices.

(iii) For $n > 2$, (2.21) cannot be reduced to a region as simple as the one characterised by Figure 1. However, when the Markov Chain is skip-free, we conjecture that a more general set conditions on μ_i than (2.21) is given by

$$\mu_i = c\sigma_i^2 + \tilde{\mu}_i,$$

where $\tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_n$. The conditions on r_i is given by $r_1 \geq \cdots \geq r_n > 0$. However it is unclear how to reconcile condition (iv) of Remark 2.25 with condition (iv') in general.

2.3.4 Extension of monotonicity results in infinite horizon problem

The monotonicity results in Section 2.3.1 - 2.3.3 can be further extended in infinite horizon. This uses an invariance property which holds for infinite horizon optimal stopping problems under the Regime Switching model.

Recall the solution of the McKean problem given in Remark 2.26 (i). This corresponds to the Regime Switching model with $n = 1$ and the jump matrix equals to the zero matrix. Let us use $u(s,1;\sigma_1^2,\mu_1,r_1,0)$ to denote the value function of the Regime Switching model for a particular choice of σ_1^2, μ_1 and r_1 in order to emphasise the value function's dependence on the parameters. Here the 0 denotes the zero-jump matrix in one dimension. We can then write

$$u(s,1;\sigma_1,\mu_1,r_1,0) = \begin{cases} K - s & : \quad s \in (0,b], \\ (K-b)\left(\dfrac{s}{b}\right)^{\gamma} & : \quad s \in (b,\infty), \end{cases}$$

where $b = -\gamma K/(1-\gamma)$, and γ stands for the negative root of the quadratic equation

$$\frac{1}{2}\sigma_1^2\gamma^2 + (\mu_1 - \tfrac{1}{2}\sigma_1^2)\gamma - r_1 = 0.$$

From this explicit solution, it is not too difficult to see that the relationship

$$u(s, 1; \sigma_1, \mu_1, r_1, 0) = u(s, 1; c\sigma_1^2, c\mu_1, cr_1, 0)$$

hold for all $c > 0$. It turns out that a multi-dimensional analogue of this property holds.

Proposition 2.27. *Let $\boldsymbol{\sigma^2} = (\sigma_1^2, \ldots, \sigma_n^2)$, $\mu = (\mu_1, \ldots, \mu_n)$ and $\mathbf{r} = (r_1, \ldots, r_n)$ be n-dimensional vectors and Q be a valid Q-matrix. We denote the value function (2.3) on page 17 with parameters $\sigma_1^2, \ldots, \sigma_n^2$, $\mu_1, \ldots, \mu_n$, $r_1, \ldots, r_n$, Q by $u(s, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q)$. In addition, assume that there is no running pay-off, i.e.*

$$h(s, i) = 0.$$

If $\tilde{g}(x, i) = g(e^x, i)$ is Lischitz in x for every $i \in \{1, \ldots, n\}$, then we have that

$$u(s, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q) = u(s, i; C\boldsymbol{\sigma^2}, C\mu, C\mathbf{r}, CQ),$$

where $C = diag(c_1, \ldots, c_n)$ is any diagonal matrix with positive diagonal entries.

Proof. Let us define $\tilde{u}(x, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q)$ by the relationship

$$\tilde{u}(x, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q) = u(e^x, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q).$$

This is just the value function written in term of log price x given by (2.4) on page 18. By Theorem 2.4, $\tilde{u}(x, i; \sigma^2, \mathbf{r}, Q)$ is the unique viscosity solution to the system of variational inequalities

$$\min(-\tilde{L}^{\boldsymbol{\sigma^2},\mu,Q}\tilde{f}(x, i; \mu, \mathbf{r}, Q) + r_i\tilde{f}(x, i; \mu, \mathbf{r}, Q),\ \tilde{f}(x, i; \mu, \mathbf{r}, Q) - \tilde{g}(x, i)) = 0, \tag{2.23}$$

with linear growth. $\tilde{L}^{\sigma^2,\mu,Q}$ is given by

$$\tilde{L}^{\boldsymbol{\sigma^2},\mu,Q}f(x, i) = \frac{1}{2}\sigma_i^2\partial_{11}f(x, i) + (\mu_i - \frac{1}{2}\sigma_i^2)\partial_1 f(x, i) - q_i f(x, i) + \sum_{j \neq i} q_{ij} f(x, j).$$

Equation (2.23) is the same as

$$\min(-c_i\tilde{L}^{\boldsymbol{\sigma^2},\mu,Q}\tilde{f}(x, i; \mu, \mathbf{r}, Q) + c_i r_i\tilde{f}(x, i; \mu, \mathbf{r}, Q),\ \tilde{f}(x, i; \mu, \mathbf{r}, Q) - \tilde{g}(x)) = 0. \tag{2.24}$$

This is because if the expression

$$-\tilde{L}^{\boldsymbol{\sigma^2},\mu,Q}\tilde{f}(x, i; \mu, \mathbf{r}, Q) + r_i\tilde{f}(x, i; \mu, \mathbf{r}, Q) \tag{2.25}$$

equals to zero, then it remains zero when multiplied by a constant. When (2.25) is non-zero,

$\tilde{f}(x,i;\mu,\mathbf{r},Q)-\tilde{g}(x,i)$ must be zero. Multiplying (2.25) by c_i does not change its sign hence (2.24) is satisfied if (2.23) is satisfied.

However, (2.24) is nothing other than

$$\min(-\tilde{L}^{C\boldsymbol{\sigma^2},C\mu,CQ}\tilde{f}(x,i;\mu,\mathbf{r},Q)+c_ir_i\tilde{f}(x,i;\mu,\mathbf{r},Q),\ \tilde{f}(x,i;\mu,\mathbf{r},Q)-\tilde{g}(x,i))=0. \quad (2.26)$$

Hence $\tilde{u}(x,i;\boldsymbol{\sigma^2},\mathbf{r},Q)$ is the unique viscosity solution to (2.26) with linear growth. By Theorem 2.4, the value function $\tilde{u}(x,i;C\boldsymbol{\sigma^2},C\mathbf{r},CQ)$ is the unique viscosity solution to (2.26) with linear growth. From this, we conclude that

$$u(s,i;\boldsymbol{\sigma^2},\mu,\mathbf{r},Q)=u(s,i;C\boldsymbol{\sigma^2},C\mu,C\mathbf{r},CQ).$$

□

Remark 2.28. (i) In the proof of the Proposition 2.27, we have shown the variational inequality satisfied by the value function $u(s,i;\boldsymbol{\sigma^2},\mu,\mathbf{r},Q)$ and the variational inequality satisfied by the value function $u(s,i;C\boldsymbol{\sigma^2},C\mu,C\mathbf{r},CQ)$ are the same. By uniqueness of the solution, the solution must coincide.

There are two alternative way of proving Proposition 2.27. Both of which are more complex comparing to the method we have used.

The first approach is to use Theorem 2.4 in combination with 'guess and verify' Lemma. We can verify that $u(s,i;\boldsymbol{\sigma^2},\mu,\mathbf{r},Q)$ satisfies the conditions of Lemma 1.6. This is, perhaps, the most complicated approach.

The second approach is directly verifying that $u(s,i;C\boldsymbol{\sigma^2},C\mu,C\mathbf{r},CQ)$ is a viscosity solution of the variational inequality (2.23).

(ii) Proposition 2.27 does not hold in finite horizon. The value function of the finite horizon problem satisfies a variational inequality similar to (2.23), but has an extra derivative with respect to time in the PDE part. Hence multiplication by constants now changes the variational inequality.

However, if we define $u(s,i,T;\sigma^2,\mu,\mathbf{r},Q)$ as the value function of the optimal stopping problem in finite horizon. It is clear from Proposition 2.27 that

$$u(s,i,T;\sigma^2,\mu,\mathbf{r},Q)-u(s,i,T;C\boldsymbol{\sigma^2},C\mu,C\mathbf{r},CQ)\to 0 \quad \text{as } T\to\infty.$$

We can now use the Proposition 2.27 to extend Corollary 2.16 and Proposition 2.18 for the infinite horizon problem.

Proposition 2.29. *If there exists $c_1, \dots, c_n > 0$ such that the conditions of Corollary 2.16 or Proposition 2.18 are satisfied for*

$$\tilde{\sigma}_i^2 = c_i \sigma_i^2,\ \tilde{\mu}_i = c_i \mu_i,\ \tilde{r}_i = c_i r_i,\ \tilde{Q} = diag\ (c_1, \dots, c_n) Q,$$

then we have

$$u(s, i; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q) \leq u(s, i'; \boldsymbol{\sigma^2}, \mu, \mathbf{r}, Q) \quad \text{for all } i, i' \in \{1, \dots, n\} \text{ such that } i \leq i'$$

Proof. First, we observe that, by Proposition 2.27,

$$u(s, i; \sigma^2, \mu, \mathbf{r}, Q) = u(s, i; C\boldsymbol{\sigma^2}, C\mu, C\mathbf{r}, CQ) = u(s, i; \tilde{\sigma}^2, \tilde{\mu}, \tilde{\mathbf{r}}, \tilde{Q}),$$

where $C = \text{diag}(c_1, \cdots, c_n)$, $\tilde{\sigma}^2 = (\tilde{\sigma}_1^2, \dots, \tilde{\sigma}_n^2)$, $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ and $\tilde{\mathbf{r}} = (r_1, \dots, r_n)$.

If $\tilde{\sigma}^2$, $\tilde{\mu}$, $\tilde{\mathbf{r}}$ and $\tilde{Q}$ satisfy the conditions of Corollary 2.16 or Proposition 2.18, then for $i < i'$, we have that

$$u(s, i; \sigma^2, \mu, \mathbf{r}, Q) = u(s, i; \tilde{\sigma}^2, \tilde{\mu}, \tilde{\mathbf{r}}, \tilde{Q}) \leq u(s, i'; \tilde{\sigma}^2, \tilde{\mu}, \tilde{\mathbf{r}}, \tilde{Q}) = u(s, i'; \sigma^2, \mu, \mathbf{r}, Q).$$

□

We illustrate Proposition 2.29 with the following example.

Example 2.30. In this example, we examine the American put option for the risk-neutral case, where $r_i = \mu_i$. This is the most relevant case to option pricing. However, the result in this example holds for a general positive, decreasing, convex gain function with $\mu_i \neq r_i$ provided $\tilde{\mu}$ defined in Proposition 2.29 satisfies the appropriate monotonicity condition.

If we choose $c_i = \sigma_i^{-2}$, then

$$\tilde{\sigma}_i^2 = 1, \quad \tilde{r}_i = \sigma_i^{-2} r_i, \quad \tilde{q}_{ij} = \sigma_i^{-2} q_{ij}.$$

Hence if the condition

$$\frac{r_1}{\sigma_1^2} \geq \cdots \geq \frac{r_n}{\sigma_n^2}, \tag{2.27}$$

and the conditions (c1') and (c2'), (which first appeared in Corollary 2.16)

(c1') for $1 \leq i < i' < j \leq n$, $\sum_{k=j}^{n} \sigma_i^{-2} q_{ik} \leq \sum_{k=j}^{n} \sigma_{i'}^{-2} q_{i'k}$,

(c2') for $1 \leq j < i < i' \leq n$, $\sum_{k=1}^{j} \sigma_i^{-2} q_{ik} \geq \sum_{k=1}^{j} \sigma_{i'}^{-2} q_{i'k}$,

are satisfied, then by Proposition 2.18, we have that

$$u(s, i) \leq u(s, i') \quad \text{for all } i, i' \in \{1, \dots, n\} \text{ such that } i \leq i'. \tag{2.28}$$

Corollary 5.1 of [2] can be seen as a special case of this result with $r_1 = \cdots = r_n$ and the jump matrix being skip-free.

Alternatively, we can choose $c_i = r_i^{-1}$, then

$$\tilde{\sigma}_i^2 = r_i^{-1}\sigma_i^2, \quad \tilde{r}_i = 1, \quad \tilde{q}_{ij} = r_i^{-1}q_{ij}.$$

Hence, if the condition (2.27) and

(c1”) for $1 \le i < i' < j \le n$, $\sum_{k=j}^{n} r_i^{-1}q_{ik} \le \sum_{k=j}^{n} r_{i'}^{-1}q_{i'k}$,

(c2”) for $1 \le j < i < i' \le n$, $\sum_{k=1}^{j} r_i^{-1}q_{ik} \ge \sum_{k=1}^{j} r_{i'}^{-1}q_{i'k}$,

are satisfied, then again by Proposition 2.18, we have that (2.28) holds. Observe that (c1”) and (c2”) coincide with (c1’) and (c2’) when the skip-free assumption is made. c_i can be chosen to be other values, but the two choices presented here are somewhat obvious choices.

When $n = 2$, all Q-matrices are skip-free. If the condition

$$\frac{\sigma_1^2}{r_1} \le \frac{\sigma_2^2}{r_2}$$

holds, then we have that $u(s, 1) \le u(s, 2)$. Otherwise, $u(s, 1) \ge u(s, 2)$ holds. Hence, the value function is always monotone in the state variable i.

2.4 Conclusion and Discussion

In this chapter, we revisited optimal stopping problems under the Regime-Switching model. We identified the issues with the ‘guess and verify’ approach when trying to verify the validity of the free boundary approach for the perpetual American put problem. We resolved these problems by showing that satisfying the classical free-boundary problem is a necessary and sufficient condition for solving the optimal stopping problem. We discussed sufficient criteria when the solutions of this free boundary problem coincides with the value function of the optimal stopping problem.

Furthermore, using an order preserving coupling of Markov chains, we strengthened an existing result on the monotonicity of the value function under Regime Switching type models. Moreover, we proved a new condition specifically for the Regime Switching model. We compared the conditions from both results and applied them to the American put problem. In the American put case, we demonstrated that these monotonicity results are consistent with the solution of the McKean problem. In the infinite horizon case, these results can be improved further using a scaling property.

A number of questions regarding the monotonicity of the value functions are left open from our investigation. We now discuss a few of them and give some conjectures. We restrict our discussion to American put problems, but similar questions can be asked about more general gain functions.

(i) All of our monotonicity results are positive results. Subject to a permutation of the states, we have proven a number of sufficient conditions for the property $u(s,i,T) \geq u(s,i',T)$ whenever $i > i'$, where T is either finite or infinite.

It is unclear whether there exists a set of parameters $\mu, \boldsymbol{\sigma^2}, \mathbf{r}, Q$ and variables s, s', i, i', T, T' such that

$$u(s,i,T) > u(s,i',T) \quad \text{and} \quad u(s',i,T) < u(s',i',T)$$

or

$$u(s,i,T) > u(s,i',T) \quad \text{and} \quad u(s,i,T') < u(s,i',T').$$

(ii) In Example 2.30, for $n = 2$, we have shown that in the risk neutral case, the sign of

$$\frac{\sigma_1^2}{r_1} - \frac{\sigma_2^2}{r_2}$$

determine the sign of $u(s,1) - u(s,2)$. Hence, the scenario described in (i) does not occur in this case.

When $\mu_i \neq r_i$, we hypothesise that the sign of $u(s,1)-u(s,2)$ is determined by $\gamma_1-\gamma_2$. γ_i is the negative root of

$$\frac{1}{2}\sigma_i^2\gamma_i^2 + (\mu_i - \frac{1}{2}\sigma_i^2)\gamma_i - r_i = 0.$$

This is known to be true when the jump matrix is zero-matrix and in the risk neutral case. In the risk-neutral case, this is equivalent to condition (2.27). Moreover, we conjecture that this is also the case when $n > 2$, subject to some restrictions on the Q-matrix. The restriction should be similar to the ordering preserving coupling of some form (c1) and (c2) found in Proposition 2.14.

We explain why we think this conjecture is true. Let's suppose that the states are ordered such that $\gamma_1 \leq ... \leq \gamma_n$ and the jump matrix is skip-free but recurrent. Our work in this chapter gave us the intuitive understanding that we can think of $u(s,i)$ as a weighted average of n disjoint Black-Scholes model

$$\text{“}u(s,i) = \sum_{j=1}^{n} w_{ij} u_{BS}(s,j)\text{”},$$

where $u_{BS}(s,i)$ is the value function of the McKean problem with parameters σ_i, μ_i

and r_i. w_{ij} is the weight. For a particular value of i, w_{ij} is largest for j closest to i. Hence, subject to some restrictions on the jump matrix, we expect $u(s,i)$ to have the same ordering as $u_{BS}(s,i)$. Although this weighted average relationship is not correct, we think the intuition nevertheless makes sense.

2.5 Chapter Appendix

Coupling construction for Proposition 2.14 We give the explicit method for constructing such a coupling but omit the verification details.

We construct a Markov Chain (Y_t, Y'_t) on the state space E of the form

$$E = \{(y_i, y_{i'}) : 1 \le i \le i' \le n\} \subset \mathcal{S} \times \mathcal{S} \tag{2.29}$$

By the definition of E, it is straightforward to see that $Y_t \le Y'_t$ for all $t \ge 0$.

Let Q be the generator matrix of a Markov Chain on the state space $\mathcal{S}$. Let Λ be the generator matrix of a Markov Chain on the state space E, where E is defined by (2.29). Λ defines a coupling of processes with Q-matrix Q if the following conditions hold.

(i) For all i, i', j,

$$\sum_{k'=j}^{n} \lambda_{(i,i')(j,k')} = q_{ij}. \tag{2.30}$$

(ii) For all i, i', j',

$$\sum_{k=1}^{j'} \lambda_{(i,i')(k,j')} = q_{i'j'}. \tag{2.31}$$

We define a pair of process (Y_t, Y'_t) living on the state space E via a generator matrix Λ. Let $\lambda_{(i,i')(j,j')}$ denote jump rate from (i,i') and (j,j') for $(i,i') \neq (j,j')$. $\lambda_{(i,i')(i,i')}$ is total rates leaving the state (i,i'). We define $\lambda_{(i,i')(j,j')}$ in 9 steps. The table below summarises all possible arrangements of i, j, i', j' such that $j \le j'$, $i < i'$.

(1) In this table, we do not consider the case $i = i'$. This occurs after the Y and Y' first meet. This is covered by case (s1) in the list.

(2) In this table, the cases $i > i'$ and $j > j'$ do not occur by the definition of E.

(3) There is an example on page 50 illustrating how the coupling is constructed.

The steps are given below:

(s1) For $1 \le i = i' \le n$,

$$\lambda_{(i,i)(j,j')} = \begin{cases} q_{ij} & \text{for } j = j', \\ 0 & \text{for } j \neq j'. \end{cases} \tag{2.32}$$

	$i = j$	$i > j$	$i < j$
$i' = j'$	Total rate Case (s9)	$j < i < i' = j'$ Case (s7)	$i < j \le i' = j'$ Case (s2)
$i' > j'$	$i = j \le j' < i'$ Case (s3)	(a) $j < i \le j' < i'$ Case (s8) (b) $j \le j' < i < i'$ Case (s6)	$i < j \le j' < i'$ Case (s8)
$i' < j'$	$i = j < i' < j'$ Case (s5)	$j < i < i' < j'$ Case (s3)	(a) $i < j \le i' < j'$ Case (s8) (b) $i < i' < j \le j'$ Case (s4)

Table 2: Table showing all possible arrangements of i, i', j, j' when $i \neq i'$

(s2) For $1 \le i < j \le i' \le n$

$$\lambda_{(i,i')(j,j')} = \begin{cases} q_{ij} & \text{for } j' = i', \\ 0 & \text{for } j' \neq i'. \end{cases} \tag{2.33}$$

(s3) For $1 \le i \le j' < i' \le n$

$$\lambda_{(i,i')(j,j')} = \begin{cases} q_{i'j'} & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases} \tag{2.34}$$

(s4) For $1 \le i < i' < j \le j' \le n$, define

$$J'^{-}(j, i, i') = \max\left\{ j' : \sum_{k'=j'}^{n} q_{i'k'} \ge \sum_{k=j}^{n} q_{ik} \right\} \tag{2.35}$$

and

$$J'^{+}(j, i, i') = \max\left\{ j' : \sum_{k'=j'}^{n} q_{i'k'} \ge \sum_{k=j+1}^{n} q_{ik} \right\}. \tag{2.36}$$

By condition (c1), it follows that $j \le J'^{-}(j, i, i') \le J'^{+}(j, i, i')$

(a) If $J'^{-}(j, i, i') = J'^{+}(j, i, i')$,

$$\lambda_{(i,i')(j,j')} = \begin{cases} q_{ij} & \text{for } j' = J'^{-}(j, i, i'), \\ 0 & \text{for } j' \neq J'^{-}(j, i, i'). \end{cases} \tag{2.37}$$

(b) If $J'^{-}(j,i,i') < J'^{+}(j,i,i')$,

$$\lambda_{(i,i')(j,j')} = \begin{cases} \sum\limits_{k=J'^{+}(j,i,i')}^{n} q_{i'k} - \sum\limits_{k=j+1}^{n} q_{ik} & \text{for } j' = J'^{+}(j,i,i'), \\ \sum\limits_{k=j}^{n} q_{ik} - \sum\limits_{k=J'^{-}(j,i,i')+1}^{n} q_{i'k} & \text{for } j' = J'^{-}(j,i,i'), \\ q_{i'j'} & \text{for } J'^{-}(j,i,i') < j' < J'^{+}(j,i,i'), \\ 0 & \text{for } j' < J'^{-}(j,i,i') \text{ or } j' > J'^{+}(j,i,i'). \end{cases} \tag{2.38}$$

(s5) For $1 \le j = i < i' < j' \le n$, define $J'^{*}(i,i')$ by

$$J'^{*}(i,i') = \max\left\{ j' : \sum_{k'=j'}^{n} q_{i'k'} \ge \sum_{k=i'+1}^{n} q_{ik} \right\} = J'^{-}(i'+1,i,i'),$$

then

$$\lambda_{(i,i')(j,j')} = \begin{cases} \sum\limits_{k'=J'^{*}(i,i')}^{n} q_{i'k'} - \sum\limits_{k=i'+1}^{n} q_{ik} & \text{for } j' = J'^{*}(i,i') \\ q_{i'j'} & \text{for } i' < j' < J'^{*}(i,i') \\ 0 & \text{for } j' > J'^{*}(i,i'). \end{cases} \tag{2.39}$$

(s6) For $1 \le j' < i < i' \le n$, define

$$J^{-}(j',i,i') = \min\left\{ j : \sum_{k=1}^{j} q_{ik} \ge \sum_{k'=1}^{j'-1} q_{i'k'} \right\}. \tag{2.40}$$

and

$$J^{+}(j',i,i') = \min\left\{ j : \sum_{k=1}^{j} q_{ik} \ge \sum_{k'=1}^{j'} q_{i'k'} \right\} \tag{2.41}$$

By condition (c2), it follows that $j \ge J^{+}(j',i,i') \ge J^{-}(j',i,i')$ and observe that fact

$$J^{-}(j'+1,i,i') = J^{+}(j',i,i') \quad \text{for } j' < i-1 \tag{2.42}$$

(a) If $J^{-}(j',i,i') = J^{+}(j',i,i')$,

$$\lambda_{(i,i')(j,j')} = \begin{cases} q_{i'j'} & \text{for } j = J^{-}(j',i,i'), \\ 0 & \text{otherwise.} \end{cases} \tag{2.43}$$

(b) If $J^-(j',i,i') < J^+(j',i,i')$,

$$\lambda_{(i,i')(j,j')} = \begin{cases} \sum_{k=1}^{J^-(j',i,i')} q_{ik} - \sum_{k'=1}^{j'-1} q_{i'k'} & \text{for } j = J^-(j',i,i'), \\ \sum_{k'=1}^{j'} q_{i'k'} - \sum_{k=1}^{J^+(j',i,i')-1} q_{ik} & \text{for } j = J^+(j',i,i'), \\ q_{ij} & \text{for } J^-(j',i,i') < j < J^+(j',i,i'), \\ 0 & \text{for } j < J^-(j',i,i') \text{ or } j > J^+(j',i,i'). \end{cases} \tag{2.44}$$

(s7) For $1 \leq j < i < i' = j' \leq n$, define $J^*(i,i')$ by

$$J^*(i,i') = \max\left\{j : \sum_{k=1}^{j} q_{ik} \geq \sum_{k=1}^{i-1} q_{i'k}\right\} = J^-(i-1,i,i'),$$

then

$$\lambda_{(i,i')(j,i')} = \begin{cases} \sum_{k=1}^{J^*(i,i')} q_{ik} - \sum_{k'=1}^{i-1} q_{i'k'} & \text{for } j = J^*(i,i') \\ q_{ij} & \text{for } J^*(i,i') < j < i \\ 0 & \text{for } j < J^*(i,i'). \end{cases} \tag{2.45}$$

(s8) For all combinations of (i,i') and (j,j') not defined in steps (s1) - (s7), we have

$$\lambda_{(i,i')(j,j')} = 0 \quad \text{for } (i,i') \neq (j,j').$$

(s9) For $(i,i') = (j,j')$,

$$\lambda_{(i,i')(j,j')} = \begin{cases} -\sum_{k=1}^{i'} q_{ik} - \sum_{k'=i}^{n} q_{i'k'} & \text{for } i \neq i' \\ q_{ii} & \text{for } i = i' \end{cases} \tag{2.46}$$

It remains to check the following for Λ to be a coupling of Q:

- the steps (s1) - (s9) define a valid Q-matrix,
- the conditions (2.30) and (2.31) are satisfied.

This can be done but we omit the proof.

Coupling example for Proposition 2.14

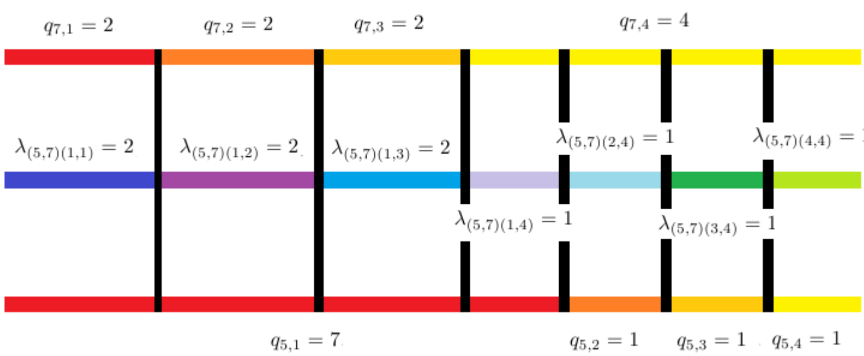
We assume the matrix satisfies (c1) and (c2) found on page 30. We illustrate how to construct a coupling for one pair of (i, i'). For $i = 5$, $i' = 7$, consider the example with

$$q_{5,1} = 7,\ q_{5,2} = 1,\ q_{5,3} = 1,\ q_{5,4} = 1,$$
$$q_{7,1} = 2,\ q_{7,2} = 2,\ q_{7,3} = 2,\ q_{7,4} = 4,$$

then it is clear that (c1) on page is satisfied for the pair $(i, i') = (5, 7)$. Following (s6) - (s7), we calculate $\lambda_{(5,7)(j,j')}$ for $j \leq j' < 5$ as follows:

$$\lambda_{(5,7)(1,1)} = 2,\ \lambda_{(5,7)(1,2)} = 2,\ \lambda_{(5,7)(1,3)} = 2,\ \lambda_{(5,7)(1,4)} = 1,$$
$$\lambda_{(5,7)(2,4)} = 1,\ \lambda_{(5,7)(3,4)} = 1,\ \lambda_{(5,7)(4,4)} = 1.$$

In the diagram below, we use the length of orange, red and yellow lines to denote the size of jump rates from $\{y_5, y_7\}$ to $\{y_1, y_2, y_3, y_4\}$. The other coloured lines denote the jump rates of the coupled chain.



Proof of Lemma 2.21

We wish to prove
$$\lim_{l \to \infty} U(s, i, T, 2^l) = u(s, i, T).$$
Observe that for $l' > l$, we have

$$U(s, i, T, 2^l) \leq U(s, i, T, 2^{l'}) \leq u(s, i, T).$$

This means the limit $\lim_{l \to \infty} U(s, i, T, 2^l)$ exists as it is the limit of a bounded monotone sequence. Since the time horizon is finite, there is a stopping time τ such that

$$u(s, i, T) = \mathbb{E}\left[e^{-\int_0^\tau r(I_t^i)\mathrm{d}t} g(S_\tau^{s,i}, I_\tau^i)\right].$$

Define the set $T_l = \{2^{-l}T, 2 \cdot 2^{-l}T, ..., (2^l - 1) \cdot 2^{-l}T, T\}$ and $\tau^l = \inf\{t \in T_l : t \geq \tau\}$. Note that $\tau^l \downarrow \tau$ almost surely as $l \to \infty$ and (S, I) is right-continuous. Hence, we have

$$
\begin{aligned}
&u(s,i,T) - U(s,i,T,2^l) \\
&\quad \leq \lim_{l\to\infty} \mathbb{E}\left[e^{-\int_0^{\tau} r(I_t^i)\mathrm{d}t} g(S_\tau^{s,i}, I_\tau^i) - e^{-\int_0^{\tau^l} r(I_t^i)\mathrm{d}t} g(S_{\tau^l}^{s,i}, I_{\tau^l}^i)\right] \\
&\quad = \mathbb{E} \lim_{m\to\infty} \left[e^{-\int_0^{\tau} r(I_t^i)\mathrm{d}t} g(S_\tau^{s,i}, I_\tau^i) - e^{-\int_0^{\tau^l} r(I_t^i)\mathrm{d}t} g(S_{\tau^l}^{s,i}, I_{\tau^l}^i)\right] \\
&\quad \to 0,
\end{aligned}
$$

where the limit exchange is permitted by dominated convergence theorem. (See Condition (2.5) on page 18.)

3 On Trading American Put Options with Interactive Volatility

This chapter is based on a joint paper with Dr Sigurd Assing. Part of this paper discusses the Regime-Switching model. This part is placed in Chapter 2.

3.1 Introduction and Results

This chapter deals with an optimal stopping problem which is motivated by option trading.

First, we introduce a simple short term model for the price of an asset which is able to capture some aspects of the so-called leverage effect, and, second, under such a model, we calculate the value and exercise time of a perpetual American put written on this asset.

The leverage effect refers to the phenomenon that, typically, decreasing asset prices are accompanied by rising volatility. We will not argue about whether the leverage effect is a true phenomenon or not. We rather treat it as an observed phenomenon which has been discussed in many papers since the mid 1970s when Black, [9], gave a well received macroeconomic explanation. Since this effect has been observed, risk-seeking market participants might want to take advantage of it.

However, other effects can superimpose a possible leverage effect. For example, a decreasing stock price after a negative earning report usually goes along with falling volatility as uncertainty decreases after an announced event. Hence, the decision to bet on a combination of falling prices and rising volatility requires a careful analysis of relevant market conditions which is left to the acting market participants.

The market participant we have in mind is an option trader who has made this decision and plans to go long on an American put. The rationale behind going long on an American put when betting on a leverage effect is twofold; falling prices and increasing volatility would both raise put prices. But, if the trader wants to understand the risk of such a betting strategy before entering the trade, they should create a model for the price $(S_t,\, t \geq 0)$ of the asset underlying the American put which, first, is simple enough, second, is able to capture key features of the trader's preferences for the future and, third, has enough parameters to control the probabilities of different scenarios of future prices.

The model we propose can heuristically be described as follows:

- the price S_t, $t \geq 0$, behaves like a geometric Brownian motion with volatility parameter σ_0 and trend μ_0 until it hits a critical level $s_0 \ll s$ where s is the present price;
- after hitting the critical level, the volatility parameter steps up to σ_1 and the trend of the stock changes to μ_1;

- this 'excited' state lasts for a period of length T which is exponentially distributed with rate λ;
- finally the price is frozen at its value taken when the exponential time T has expired.

Note that the above bullet points characterise a type of stochastic volatility model which has not been discussed in the literature, yet. We call such a stochastic volatility *interactive volatility* to emphasise its dependence on hitting times of the price process.

Remark 3.1. (i) The above model supposes $S_t = S_{t\wedge(\tau_{s_0}+T)}$ for $t \geq 0$ where

$$\tau_a \stackrel{\text{def}}{=} \inf\{t \geq 0 : S_t \leq a\} \tag{3.1}$$

for given price levels a. The reason for freezing the price at $\tau_{s_0} + T$ is that this time span is considered the time horizon of the trade. Studying a <u>perpetual</u> American put under this model easily reveals that the put is exercised at the random time $\tau_{s_0} + T$, at the latest—see Remark 3.3(ii).

(ii) The notation $s_0 \ll s$ is used to emphasise that the difference $s - s_0$ between the present price of the asset and the critical level should be chosen big enough. The size of $s - s_0$ determines the strength of the market's drop which causes the regime change from volatility σ_0 to σ_1 according to the leverage effect.

(iii) A reasonable choice for σ_0 would be the implied volatility at present time of the <u>traded</u> American put the trader wants to go long on. Now recall that

$$\log S_t \;=\; \log s \,+ (\mu_0 - \frac{\sigma_0^2}{2})\, t \,+ \sigma_0 \overbrace{B_t}^{\text{Brownian motion}}$$

is assumed to hold for $t \in [0, \tau_{s_0})$. Hence, for fixed σ_0, the choice of μ_0 determines the distribution of the hitting time τ_{s_0}. To meet the preferences of the trader of a market-fall in the near future, μ_0 should be chosen sufficiently small to decrease the probability of large values of τ_{s_0}. But, the trader should also analyse the optimal stopping problem for larger values of μ_0, that is, they should analyse their position under the assumption they are wrong and the probability of a market-fall in the near future is rather small.

(iv) For $t \in [\tau_{s_0}, \tau_{s_0} + T)$, which is the final period of the trade, the trader assumes

$$\log S_t \;=\; \log s_0 \,+ (\mu_1 - \frac{\sigma_1^2}{2})(t - \tau_{s_0}) + \,\sigma_1(B_t - B_{\tau_{s_0}}).$$

The choice of the parameters σ_1, μ_1 reflects the trader's view on the strength of the regime change triggered by the leverage effect, and hence this choice is more or less subject to both the trader's experience and their understanding of the market's history.

(v) Using an exponential time T for modelling the time span of the impact of the leverage effect keeps the model simple enough. It is also assumed that T is independent of what has happened before τ_{s_0}. The parameter λ should be big enough to ensure that, on average, the time span of the new volatility regime is of the order of days and not weeks. If both τ_{s_0} and T are on average rather short then the whole trade's time horizon is likely to be less than the time to maturity of traded American puts.

(vi) Following the suggestions made in items (iii),(iv) above, the trader's reasoning behind choosing μ_0, μ_1 has nothing to do with the market's rate of interest during the time span of their trade. Thus, the model's underlying probability measure should be considered a guess of the real-world measure rather than a pricing measure. Working out the optimal exercise time of the perpetual American put in the context of this model gives the trader an indication of when to exit a trade they entered in accordance with their own preferences for the future. The value of the put under the model is mainly used for finding the optimal exercise time, and should NOT be confused with the price of a traded put.

(vii) Our analysis can be used to motivate the choice of a traded American put with reasonable strike level and time to maturity for the purpose of betting on a combination of falling prices and rising volatility—see Section 3.3 for the details.

The proposed model has features of a Markov chain regime switching volatility model as the excited state, when the volatility is σ_1, lasts for an exponential time. But, at the end of this exponential time, instead of moving into a state which corresponds to another volatility level, the Markov chain moves into an absorbing state. So, for the second and final period of the trade, the model can be regarded as a degenerated Markov chain regime switching volatility model. We will comment on Markov chain regime switching volatility models in Remark 3.7(ii) below.

For the initial period of the trade, the model is different to a Markov chain regime switching volatility model as the system does not enter the excited state following the move of a Markov chain. Instead, it enters the excited state according to if the price of the asset has fallen to the critical level s_0 or not, that is, according to how the price of the asset has behaved in the past.

To achieve Markovianity, we add a process $(Y_t,\, t \geq 0)$ for screening whether the price S_t has already fallen to the critical level s_0 or not. To fully describe the dynamics of S_t, we also add a process $(\eta_t,\, t \geq 0)$ which is an absorbing Markov chain screening the length of the excitation.

Technically, we work with a strong Markov process $(S,Y,\eta)=(S_t,Y_t,\eta_t,t\geq 0)$ on a family of probability spaces $(\Omega,\mathcal{F},\mathbf{P}_{s,y,i},(s,y,i)\in A)$, where

$$A \stackrel{\text{def}}{=} \Big[\ (s_0,\infty)\times\{0\}\times\{1\}\ \Big]\cup\Big[\ (0,\infty)\times\{1\}\times\{0,1\}\ \Big]$$

is considered a subset of the topological space $(0,\infty)\times\{0,1\}\times\{0,1\}$ equipped with the product topology.

The generator of this process is formally defined by

$$\begin{array}{lcll} Lf(s,0,1) & = & \mu_0 s\partial_1 f(s,0,1)+\frac{1}{2}\sigma_0^2 s^2\partial_1^2 f(s,0,1) & \text{for } s\in(s_0,\infty), \\ Lf(s,1,1) & = & \mu_1 s\,\partial_1 f(s,1,1)+\frac{1}{2}\sigma_1^2 s^2\partial_1^2 f(s,1,1)+\lambda\left[f(s,1,0)-f(s,1,1)\right] & \text{for } s\in(0,\infty), \\ Lf(s,1,0) & = & 0 & \text{for } s\in(s,\infty). \end{array}$$

It is considered an unbounded operator on the space $C_b(\bar{A})$ of all bounded continuous functions on $\bar{A}$ whose domain consists of all $f\in C_b(\bar{A})$ satisfying both $f(s_0,1,1)-f(s_0,0,1)=0$ and $Lf\in C_b(\bar{A})$ where the latter condition is understood in the sense of Schwartz distributions.

Remark 3.2. (i) The condition $f(s_0,1,1)-f(s_0,0,1)=0$ is a discrete Neumann boundary condition, and this boundary condition establishes an interaction between the states $(s_0,0,1)$ and $(s_0,1,1)$ leading to a jump of the process Y_t when the price S_t reaches s_0.

(ii) As a consequence, for all $s>s_0$, under $\mathbf{P}_{s,0,1}$, it holds that $Y_t=\mathbf{1}_{[\tau_{s_0},\infty)}(t)$, $t\geq 0$, and $\eta_t=1$, $t\leq\tau_{s_0}$, whereas, for all $s>0$, under $\mathbf{P}_{s,1,1}$, it holds that $Y_t=1$, $t\geq 0$, and $(\eta_t,\,t\geq 0)$ is an independent two-states continuous-time Markov chain starting from one and absorbed at zero with rate λ.

(iii) Combining (ii) and $Lf(s,1,0)=0$ for $s>0$ yields

$$\mathbf{P}_{s,y,i}\Big(\ \{\ S_t\ =\ S_{t\wedge\tau_{\eta,0}}\ \text{for all } t\geq 0\}\ \Big)=1\quad\text{for all } (s,y,i)\in A,$$

where

$$\tau_{\eta,0}\stackrel{\text{def}}{=}\inf\{t\geq 0:\eta_t=0\},\tag{3.2}$$

that is, $\tau_{\eta,0}$ plays the role of what was called $\tau_{s_0}+T$ in Remark 3.1(i).

(iv) Taking into account the other defining properties of the generator L, the S-component of the process (S,Y,η) started at $s>s_0$ has, under $\mathbf{P}_{s,0,1}$, the same law as the price process discussed in items (iii) and (iv) of Remark 3.1. Note that we could have worked with the process (S,Y) killed at rate λ after the jump of Y, instead. But, as explained in Remark 3.7(ii) below, using an extra component like η has the advantage that we can apply results on optimal stopping in the context of Markov chain regime switching models.

(v) Since (S, Y, η) is strong Markov, the filtration $(\mathcal{F}_t,\, t \geq 0)$ generated[1] by (S, Y, η) is right-continuous, and, by obvious reasons, this filtration coincides with the smallest right-continuous filtration which contains the universal augmentation of $(\sigma(S_u : u \leq t),\, t \geq 0)$.

All in all, we have established a probability model for the prospective prices of an asset which induces the wanted features laid out in the four bullet points on page 53.

Next, under this model, we will study the value and optimal exercise time of a perpetual American put contract written on this asset. Using our probability model, such a put's value function takes the form

$$V(s, y, i) \stackrel{\text{def}}{=} \sup_{\tau \geq 0} \mathbf{E}_{s,y,i}[e^{-\alpha\tau}(K - S_\tau)^+] \quad \text{for} \qquad (s, y, i) \in A, \tag{3.3}$$

where the supremum is taken over all stopping times with respect to the filtration $(\mathcal{F}_t,\, t \geq 0)$.

Remark 3.3. (i) The discount rate α refers to the rate of return of an investment the trader considers more or less riskless during the time interval of the trade. As explained in items (iii) and (v) of Remark 3.1, under the future preferences of the trader, this time interval is supposed to be rather short on average, and hence choosing α to be constant is a good approximation.

(ii) Note that

$$V(s, y, i) = \sup_{\tau \leq \tau_{\eta,0}} \mathbf{E}_{s,y,i}[e^{-\alpha\tau}(K - S_\tau)^+]$$

because $S_t = S_{t \wedge \tau_{\eta,0}}$, $t \geq 0$, implies $e^{-\alpha\tau}(K - S_\tau)^+ \leq e^{-\alpha\tau_{\eta,0}}(K - S_{\tau_{\eta,0}})^+$ on $\tau \geq \tau_{\eta,0}$. Hence, under our model, the perpetual put should be exercised at $\tau_{\eta,0} = \tau_{s_0} + T$, at the latest.

First recall the results for perpetual American put options in the context of geometric Brownian motion.

Theorem 3.4. *Given on a family of probability spaces $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}_s, s > 0)$, let $(\tilde{S}_t,\, t \geq 0)$ be the Feller process whose generator is the closure of*

$$\tilde{L}f = \mu_0 s f' + \frac{1}{2}\sigma_0^2 s^2 f'', \quad f \in C_b^2((0, \infty)).$$

[1] Here, *'filtration generated by* (S, Y, η)*'* refers to the universal augmentation of the filtration $(\sigma(S_u, Y_u, \eta_u : u \leq t),\, t \geq 0)$ see Section 2.7.B of [41] for a good account on universal filtrations.

Then, the value function

$$\tilde{V}(s) \stackrel{def}{=} \sup\Big\{\tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}}(K-\tilde{S}_{\tilde{\tau}})^+] : \tilde{\tau} \text{ stopping time with respect to } (\tilde{S}_t,\, t\geq 0)\Big\}$$

is given by

$$\tilde{V}(s) = \begin{cases} K-s & : \quad s\in(0,b_0], \\ (K-b_0)\left(\dfrac{s}{b_0}\right)^{\gamma^-} & : \quad s\in(b_0,\infty), \end{cases} \tag{3.4}$$

where $b_0 = -\gamma^- K/(1-\gamma^-)$, and γ^- stands for the negative root of the quadratic equation

$$\frac{1}{2}\sigma_0^2\gamma^2 + (\mu_0 - \frac{1}{2}\sigma_0^2)\gamma - \alpha = 0. \tag{3.5}$$

Remark 3.5. (i) The above value function $\tilde{V}$ satisfies

$$0 \;=\; \mu_0 s\tilde{V}'(s) + \frac{1}{2}\sigma_0^2 s^2\tilde{V}''(s) - \alpha\tilde{V}(s) \quad \text{for } s > b_0$$

subject to

$$\tilde{V}(b_0) = K - b_0, \quad \tilde{V}'(b_0) = -1, \quad \lim_{s\to\infty}\tilde{V}(s) = 0.$$

(ii) If $\mu_0 - \frac{1}{2}\sigma_0^2 > 0$, then there is no (finite) optimal stopping time at which the value function $\tilde{V}$ can be attained. But $\tilde{\tau}_{b_0}$ is a Markov time at which $\tilde{V}$ given in Theorem 3.4 is attained when setting $\tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{b_0}}(K-\tilde{S}_{\tilde{\tau}_{b_0}})^+]$ to be zero on $\{\tilde{\tau}_{b_0}=\infty\}$. In all further cases below, attaining a value function at a possibly infinite Markov time will be understood as above, since all considered value functions vanish at infinity.

The next theorem presents the main result of this chapter.

Theorem 3.6. *Recall (3.1), (3.2), and Remark 3.1(ii) for the purpose of s_0, and Remark 3.5(ii) for the meaning of b_0. Let γ^+ (γ^-) denote the positive (negative) root of equation (3.5). The following cases completely describe the value function given by (3.3).*

(i) In the trivial case,

$$V(s,1,0) = (K-s)^+ \quad \textit{for all } s > 0,$$

and the optimal stopping time is 0.

(ii) Let β^+ (β^-) denote the positive (negative) root of the quadratic equation

$$\frac{1}{2}\sigma_1^2\beta^2 + (\mu_1 - \frac{1}{2}\sigma_1^2)\beta - (\alpha+\lambda) = 0. \tag{3.6}$$

Then, there exists a smooth function $h : (0,\infty) \to \mathbb{R}$ such that

$$V(s,1,1) = \begin{cases} K - s & : \quad s \in (0, b_1], \\ c_1 s^{\beta^+} + c_2 s^{\beta^-} + h(s) & : \quad s \in (b_1, K], \\ d_2 s^{\beta^-} & : \quad s \in (K, \infty), \end{cases}$$

where the coefficients c_1, c_2, d_2 and the stopping level b_1 are obtained by solving the equations (3.10) on page 62. The finite optimal stopping time is $\tau_{b_1} \wedge \tau_{\eta,0}$.

(iii) If one of the conditions

(a) $b_0 > s_0$ and $V(s_0,1,1) \geq (K - b_0)(s_0/b_0)^{\gamma^-}$,

(b) $b_0 \leq s_0$ and $V(s_0,1,1) > (K - s_0)^+$,

(c) $b_0 \leq s_0 < K$ and $V(s_0,1,1) = (K - s_0)$,

is satisfied, then

$$V(s,0,1) = V(s_0,1,1)\left(\frac{s}{s_0}\right)^{\gamma^-} \quad \text{for} \quad s > s_0.$$

This value function is attained at the possibly infinite Markov time $\tau_{b_1} \wedge \tau_{\eta,0}$, if (a),(b), and τ_{s_0}, if (c).

(iv) If $b_0 > s_0$ and $K - s_0 \leq V(s_0,1,1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, then

$$V(s,0,1) = \begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : \quad s \in (s_0, b_*), \\ K - s & : \quad s \in [b_*, b_0] \cap (s_0, \infty), \\ (K - b_0)\left(\dfrac{s}{b_0}\right)^{\gamma^-} & : \quad s \in (b_0, \infty), \end{cases}$$

where[2] $b_* = s_0$ if $K - s_0 = V(s_0,1,1)$, and e_1^*, e_2^*, b_* as obtained in the proof of Lemma 3.10 on page 68, otherwise. The value function is attained at the possibly infinite Markov time $\tau_{[b_*,b_0],0} \wedge \tau_{b_1} \wedge \tau_{\eta,0}$ where

$$\tau_{[a,b],0} \overset{def}{=} \inf\{t \geq 0 : S_t \in [a,b], Y_t = 0\}$$

for levels $0 < a < b$.

Remark 3.7. (i) The contribution of this chapter consists in the two non-trivial cases (iii) and (iv) but also in the review of the known case (ii)—see Remark 3.7(iii+iv)

[2]Here $(s_0, b_*) = \emptyset$ by definition when $b_* = s_0$.

below. Case (iv) is mathematically most interesting because of an unexpectedly disconnected continuation region. As explained in Remark 3.1(vi), only the optimal stopping times are of true relevance for the trader. Note that there are no further sub-cases than those mentioned under (iii) and (iv). The restriction to $s_0 < K$ in case (iii)(c) follows from the fact that $V(s_0, 1, 1) > 0$ if $s_0 \geq K$. We will discuss numerical examples for all sub-cases in Section 3.3.

(ii) Both cases (iii) and (iv) are based on case (ii), that is, on the solution of the optimal stopping problem under the measures $\mathbf{P}_{s,1,1}$, $s > 0$. Under these measures, the price process $(S_t,\, t \geq 0)$ satisfies the stochastic differential equation

$$\mathrm{d}S_t \;=\; \mu_1 \eta_t S_t \,\mathrm{d}t + \sigma_1 \eta_t S_t \,\mathrm{d}B_t$$

which is a consequence of what was discussed in Remark 3.1(iv) and (ii)+(iii) of Remark 3.2. Note that the above stochastic differential equation describes a Markov modulated geometric Brownian motion, which was covered in Chapter 2. The explicit form of the value function given in case (ii) is a special case of Guo/Zhang's result when the Markov chain is degenerated. The function $h : (0, \infty) \to \mathbb{R}$ can be any solution to the first of the two differential equations above (3.8) on page 62, for example,

$$h(s) \;=\; -\frac{\lambda s}{\alpha + \lambda - \mu_1} + \frac{\lambda K}{\alpha + \lambda} \quad \text{and} \quad h(s) \;=\; \frac{\lambda s \log s}{\alpha + \lambda + \sigma_1^2/2} + \frac{\lambda K}{\alpha + \lambda}$$

for $\alpha + \lambda \neq \mu_1$ and $\alpha + \lambda = \mu_1$, respectively.

(iii) To prove that the explicit expression given for $V(\cdot, 1, 1)$ in case (ii) is indeed the value function requires a verification argument. Denoting this explicit expression by $V^*(\cdot, 1, 1)$, the standard method of verification would be to verify the properties (v1),(v2),(v3) as given at the beginning of Section 3.2.3, but with respect to the measures $\mathbf{P}_{s,1,1}$, $s > 0$.

3.2 Proofs

Both cases (i) and (ii) of Theorem 3.6 assume $y = 1$ which leads to a special version of the results obtained in [33] where American puts were priced in the context of a two-states Markov chain volatility model. In our case, the Q-matrix of the corresponding Markov chain is degenerate.

Therefore, in the next section, we only sketch the proof of case (ii). But we give enough details to put the notation used in Theorem 3.6(ii) into context. However, recall Remark 3.7(iii) where we explained that the verification of the value function in [33] was incomplete. The corresponding details can be found in the Appendix.

3.2.1 Proof of Theorem 3.6 (i) and (ii)

For all $s > 0$, and any stopping time τ,

$$\mathbf{E}_{s,1,i}[e^{-\alpha\tau}(K - S_\tau)^+] \le \mathbf{E}_{s,1,i}[e^{-\alpha(\tau\wedge\tau_{\eta,0})}(K - S_{\tau\wedge\tau_{\eta,0}})^+] \quad \text{for } (s,i) \in (0,\infty)\times\{0,1\}$$

since the process $(S_t, t \ge 0)$ is stopped at $\tau_{\eta,0}$. Hence $V(s,1,0) = (K-s)^+$ with optimal stopping time 0 proving (i).

For showing (ii), recall that $Y_t = 1$ for all $t \ge 0$ a.s. when starting the dynamics from any $s > 0$, $y = i = 1$. Now, assume that the stopping region takes the form

$$(0, b_1] \times \{1\} \times \{1\} \cup (0,\infty) \times \{1\} \times \{0\}$$

when starting from $s > 0$, $y = i = 1$ where b_1 is an unknown stopping level. Then, by [57, Theorem 2.4] for example, if V is lower semi-continuous, the value function $V(s,1,1)$ would be attained at

$$\tau^* = \tau_{\eta,0} \wedge \tau_{b_1}, \tag{3.7}$$

and $(e^{-\alpha(\tau^*\wedge t)}V(S_{t\wedge\tau^*}, Y_{t\wedge\tau^*}, \eta_{t\wedge\tau^*}),\, t \ge 0)$ would be a $\mathbf{P}_{s,1,1}$-martingale.

We want to use this martingale property to derive equations for both V and b_1. Assume for now that V has even more regularity and a generalised Itô's formula (see Remark 3.9 below) can be applied to obtain

$$e^{-\alpha(t\wedge\tau^*)}V(S_{t\wedge\tau^*}, Y_{t\wedge\tau^*}, \eta_{t\wedge\tau^*}) = V(S_0, Y_0, \eta_0) + \int_0^{t\wedge\tau^*} e^{-\alpha u}(L - \alpha I)V(S_u, Y_u, \eta_u)\mathrm{d}u + M_{t\wedge\tau^*}$$

for all $t \ge 0$ a.s. where M stands for a local martingale and I denotes the identity operator.

Of course, for the above left-hand side to be a martingale, the integral on the right-hand side must vanish. Using both the specific form of L as given on page 56 and (i) proven above, a sufficient condition for this integral to vanish is

$$\begin{cases} 0 \;=\; \mu_1 s\partial_1 V(s,1,1) + \frac{1}{2}\sigma_1^2 s^2 \partial_{11} V(s,1,1) + \lambda(K-s) - (\alpha+\lambda)V(s,1,1) & \text{for } s \in (b_1, K) \\ 0 \;=\; \mu_1 s\partial_1 V(s,1,1) + \frac{1}{2}\sigma_1^2 s^2 \partial_{11} V(s,1,1) - (\alpha+\lambda)V(s,1,1) & \text{for } s \in (K,\infty) \end{cases}$$

depending on the unknown b_1 subject to the boundary and pasting conditions

$$\begin{aligned}
\lim_{s\to\infty} V(s,1,1) &= 0, \\
V(K-,1,1) &= V(K+,1,1), \\
\partial_1 V(K-,1,1) &= \partial_1 V(K+,1,1), \\
K - b_1 &= V(b_1+,1,1), \\
-1 &= \partial_1 V(b_1+,1,1),
\end{aligned} \tag{3.8}$$

where $-$ and $+$ indicate taking left and right limits at the corresponding argument, respectively.

The well-known solution of the above equation has the form

$$V(s,1,1) = \begin{cases} c_1 s^{\beta^+} + c_2 s^{\beta^-} + h(s) & \text{for } s \in (b_1, K) \\ d_1 s^{\beta^+} + d_2 s^{\beta^-} & \text{for } s \in (K, \infty) \end{cases} \tag{3.9}$$

with unknown coefficients c_1, c_2, d_1, d_2, and β^+ (β^-) being the positive (negative) root of equation (3.6). For the choice of the function h we refer to Remark 3.7(ii).

Certainly, the first of the conditions under (3.8) implies $d_1 = 0$, and the other four conditions yield

$$\begin{aligned}
c_1 K^{\beta^+} + c_2 K^{\beta^-} + h(K) &= d_2 K^{\beta^-}; \\
c_1 \beta^+ K^{\beta^+} + c_2 \beta^- K^{\beta^-} + K h'(K) &= d_2 \beta^- K^{\beta^-}; \\
K - b_1 &= c_1 b_1^{\beta^+} + c_2 b_1^{\beta^-} + h(b_1); \\
-b_1 &= c_1 \beta^+ b_1^{\beta^+} + c_2 \beta^- b_1^{\beta^-} + b_1 h'(b_1).
\end{aligned} \tag{3.10}$$

Note that the coefficients c_1, c_2, d_2 linearly depend on $b_1^{\beta^\pm}$ so that the problem comes down to solving numerically for b_1. For verification we refer to the Appendix.

This concludes the discussion of V for $y = 1$. We now turn to the cases (iii) and (iv) of Theorem 3.6 dealing with the case $y = 0$.

3.2.2 Proof of Theorem 3.6 (iii)

Recall the setup of Theorem 3.4 but also introduce

$$\tilde{V}_0(s) \overset{\text{def}}{=} \sup \left\{ \tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}}(K - \tilde{S}_{\tilde{\tau}\wedge\tilde{\tau}_{s_0}})^+] : \tilde{\tau} \text{ stopping time with respect to } (\tilde{S}_t,\, t \geq 0) \right\}$$

for all $s \geq s_0$.

Lemma 3.8. *If $b_0 \leq s_0 < K$ then*

$$\tilde{V}_0(s) = (K - s_0)\left(\frac{s}{s_0}\right)^{\gamma^-} \quad \textit{for} \quad s \geq s_0, \tag{3.11}$$

and the (possibly infinite) Markov time $\tilde{\tau}_{s_0}$ is the optimal time.

Proof. Assume $b_0 \leq s_0 < K$ and fix $s \geq s_0$. By 'guess and verify', it suffices to check that the right-hand side of (3.11) satisfies

(v1) $(K - s_0)(s/s_0)^{\gamma^-} = \tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0}<\infty\}}]$;

(v2) the process $(e^{-\alpha t}(K - s_0)(\tilde{S}_{t\wedge\tilde{\tau}_{s_0}}/s_0)^{\gamma^-}, t \geq 0)$ is a $\tilde{\mathbf{P}}_s$-supermartingale;

(v3) $(K - s_0)(s/s_0)^{\gamma^-} \geq (K - s)^+$.

For (v1), by Itô's formula,

$$\begin{aligned}
&\tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0}<\infty\}}] \\
&= \lim_{t\to\infty} \tilde{\mathbf{E}}_s[\, e^{-\alpha(\tilde{\tau}_{s_0}\wedge t)}(K - s_0)\left(\frac{\tilde{S}_{\tilde{\tau}_{s_0}\wedge t}}{s_0}\right)^{\gamma^-}] \\
&= \lim_{t\to\infty} \tilde{\mathbf{E}}_s\left[(K - s_0)\left(\frac{s}{s_0}\right)^{\gamma^-} + (K - s_0)\int_0^{\tilde{\tau}_{s_0}\wedge t} e^{-\alpha u}\,[\,(\tilde{L} - \alpha I)\left(\frac{\cdot}{s_0}\right)^{\gamma^-}](\tilde{S}_u)\,du + M_{t\wedge\tilde{\tau}_{s_0}}\right] \\
&= (K - s_0)\left(\frac{s}{s_0}\right)^{\gamma^-}
\end{aligned}$$

because $(M_t, t \geq 0)$ is a $\tilde{\mathbf{P}}_s$-martingale and the expression inside the integral vanishes.

For (v2), by the Markov Property, it suffices to prove that $\tilde{\mathbf{E}}_s[e^{-\alpha t}(\tilde{S}_{t\wedge\tilde{\tau}_{s_0}}/s_0)^{\gamma^-}] \leq (s/s_0)^{\gamma^-}$ for all $t \geq 0$ ignoring the constant $K - s_0$. But, for fixed $t \geq 0$,

$$\tilde{\mathbf{E}}_s[\, e^{-\alpha t}\left(\frac{\tilde{S}_{t\wedge\tilde{\tau}_{s_0}}}{s_0}\right)^{\gamma^-}] \leq \tilde{\mathbf{E}}_s[\, e^{-\alpha(t\wedge\tilde{\tau}_{s_0})}\left(\frac{\tilde{S}_{t\wedge\tilde{\tau}_{s_0}}}{s_0}\right)^{\gamma^-}] = \left(\frac{s}{s_0}\right)^{\gamma^-}.$$

where the last equality was already verified when proving (v1) above.

For (v3), note that

$$\tilde{V}'(b_0) = (K - b_0)\,\frac{\gamma^-}{b_0} = -1$$

as mentioned in Remark 3.5(i). Based on two arguments, we can now deduce that the derivative of $(K - s_0)(\cdot/s_0)^{\gamma^-}$ is bigger than -1 on $s \in (s_0, K)$. First, the derivative of

$(K-s_0)(\cdot/s_0)^{\gamma^-}$ is bounded below by -1 at s_0 because $b_0 \le s_0$ implies

$$\frac{\tilde{V}'(b_0)}{(K-s_0)\gamma^-/s_0} = \frac{(K-b_0)s_0}{(K-s_0)b_0} \ge 1,$$

and, second, $(K-s_0)(\cdot/s_0)^{\gamma^-}$ is convex on (s_0,K).

But, if the derivative of $(K-s_0)(\cdot/s_0)^{\gamma^-}$ is bigger than -1 on (s_0,K) and $(K-s_0)(\cdot/s_0)^{\gamma^-}$ touches $(K-\cdot)$ at s_0, then $(K-s_0)(s/s_0)^{\gamma^-} > (K-s)^+$ for all $s \in (s_0,K)$. Finally, $(K-s_0)(s/s_0)^{\gamma^-} > (K-s)^+ = 0$ for $s \in [K,\infty)$ is obvious. □

Remark 3.9. In what follows, we are going to use an easy application of Meyer's, [50], generalised Itô's formula which goes as follows: if $\phi:(0,\infty)\to\mathbb{R}$ is a function which is twice continuously differentiable, except at finitely many points $\{a_1,\dots,a_n\}$, such that

$$\phi'(a_k\pm) \stackrel{\text{def}}{=} \lim_{x\to a_k\pm}\phi'(x) \quad \text{and} \quad \phi''(a_k\pm) \stackrel{\text{def}}{=} \lim_{x\to a_k\pm}\phi''(x)$$

exist and are finite, $k=1,\dots,n$, then[3]

$$\phi(S_t) \;=\; \phi(S_0) \;+\int_0^t \phi'(S_u)\,\mathrm{d}S_u \;+\frac{1}{2}\int_0^t \phi''(S_u)\,\mathrm{d}\langle S\rangle_u \;+\sum_{k=1}^n \frac{1}{2}\,L_t(a_k)\,[\phi'(a_k+)-\phi'(a_k-)]$$

for all $t \ge 0$ a.s. where L_t stands for the local time of the continuous semimartingale $(S_t,\, t\ge 0)$. Note that both integrands in the above formula are well-defined for Lebesgue almost every u almost surely which uniquely determines the integrals.

If ϕ' is continuous then the local time terms would even vanish so that the above formula would look like the classical Itô's formula.

We now return to our problem of finding $V(s,0,1)$ for $s > s_0$. Recall that we already know $V(s,1,i)$ for all $s>0$ and $i=0,1$.

3.2.3 Proof of Theorem 3.6 (iii)(a) and (iii)(b)

Suppose that either condition (a) or condition (b) of Theorem 3.6(iii) is satisfied. Recall $\tau^* = \tau_{b_1} \wedge \tau_{\eta,0}$ and introduce

$$V^*(s,y,i) \stackrel{\text{def}}{=} \begin{cases} V(s_0,1,1)\left(\dfrac{s}{s_0}\right)^{\gamma^-} & : \quad s > s_0, y=0, i=1, \\ V(s,1,i) & : \quad s>0, y=1, i\in\{0,1\}. \end{cases}$$

Verifying, for any $s > s_0$,

(v1) $V^*(s,0,1) = \mathbf{E}_{s,0,1}[e^{-\alpha\tau^*}(K-S_{\tau^*})^+\,\mathbf{1}_{\{\tau^*<\infty\}}]$,

[3] Note that $(S_t,\, t\ge 0)$ only takes positive values.

(v2) the process $(e^{-\alpha t}\, V^*(S_t, Y_t, \eta_t),\ t \geq 0)$ is a $\mathbf{P}_{s,0,1}$ - supermartingale,

(v3) $V^*(s, 0, 1) \geq (K - s)^+$,

would imply the conclusion of Theorem 3.6 in both cases (a) and (b).

We are going to verify (v1),(v2),(v3). First observe that $V(s_0, 1, 1) > (K - s_0)^+$ holds in both cases (a) and (b). To see this in the non-trivial case (a), note that $(K - b_0)(\cdot/b_0)^{\gamma^-}$ is strictly convex on $[s_0, b_0]$ and touches $(K - s)^+$ at $s = b_0 < K$, so $V(s_0, 1, 1) \geq (K - b_0)(s_0/b_0)^{\gamma^-} > K - s_0$.

As a consequence, $(s_0, 1, 1)$ is in the continuation region with respect to the optimal stopping problem (3.3) on page 57. Since b_1 is the lower boundary of the continuation region when $y = i = 1$ we can deduce that $b_1 < s_0$, and hence, $\tau_{s_0} < \tau^*$ for $s > s_0$. Therefore

$$\begin{aligned}
&\mathbf{E}_{s,0,1}[e^{-\alpha\tau^*}(K - S_{\tau^*})^+\,\mathbf{1}_{\{\tau_{s_0}<\infty\}}] \\
=&\mathbf{E}_{s,0,1}\left[\mathbf{E}_{s,0,1}[e^{-\alpha\tau^*}(K - S_{\tau^*})^+\,\mathbf{1}_{\{\tau_{s_0}<\infty\}}|\mathcal{F}_{\tau_{s_0}}]\right] \\
=&\mathbf{E}_{s,0,1}\left[\ e^{-\alpha\tau_{s_0}}\,\mathbf{1}_{\{\tau_{s_0}<\infty\}}\,\underbrace{\mathbf{E}_{s_0,1,1}[e^{-r\tau^*}(K - S_{\tau^*})^+]}_{V(s_0,1,1)}\ \right]
\end{aligned}$$

by the strong Markov property. So, simply working out the Laplace transform of the hitting time τ_{s_0} yields (v1).

Next we verify (v2) for $s > s_0$ fixed. By Markov Property, we only need to show that

$$\mathbf{E}_{s,0,1}[e^{-\alpha t}V^*(S_t, Y_t, \eta_t)] \leq V^*(s, 0, 1) \tag{3.12}$$

for all $t \geq 0$.

Fix $t \geq 0$ and consider

$$\begin{aligned}
e^{-\alpha t}\,V^*(S_t, Y_t, \eta_t) \ &= \ e^{-\alpha t}\,V^*(S_t, Y_t, \eta_t) - e^{-\alpha(t\wedge\tau_{s_0})}\,V^*(S_{t\wedge\tau_{s_0}}, Y_{t\wedge\tau_{s_0}}, \eta_{t\wedge\tau_{s_0}}) \\
&\quad + \ e^{-\alpha(t\wedge\tau_{s_0})}\,V^*(S_{t\wedge\tau_{s_0}}, 0, 1)
\end{aligned} \tag{3.13}$$

where the last term is justified by $V^*(s_0, 1, 1) = V^*(s_0, 0, 1)$.

Now realise that, by strong Markov property,

$$\begin{aligned}
&\mathbf{E}_{s,0,1}[\,\mathbf{1}_{\{t\geq\tau_{s_0}\}}\,e^{-\alpha t}\,V^*(S_t, Y_t, \eta_t)\,] \\
= \ &\mathbf{E}_{s,0,1}\left[\,\mathbf{1}_{\{t\geq\tau_{s_0}\}}\mathbf{E}_{s,0,1}[\,e^{-\alpha t}\,V^*(S_t, Y_t, \eta_t)\,|\,\mathcal{F}_{t\wedge\tau_{s_0}}\,]\,\right] \\
= \ &\int \mathbf{1}_{\{t\geq\tau_{s_0}(\omega)\}}\,e^{-\alpha\tau_{s_0}(\omega)}\ \underbrace{\mathbf{E}_{s_0,1,1}[\,e^{-\alpha(t-\tau_{s_0}(\omega))}\,V^*(S_{t-\tau_{s_0}(\omega)}, Y_{t-\tau_{s_0}(\omega)}, \eta_{t-\tau_{s_0}(\omega)})\,]}_{\leq\ V^*(s_0,1,1) \text{ from case (ii)}}\,\mathbf{P}_{s,0,1}(\mathrm{d}\omega),
\end{aligned}$$

and, since the difference on the right-hand side of (3.13) equals

$$\Big[\, e^{-\alpha t}\, V^*(S_t, Y_t, \eta_t) - e^{-\alpha\tau_{s_0}}\, V^*(s_0, 1, 1)] \,\Big] \times \mathbf{1}_{\{t\geq\tau_{s_0}\}},$$

we obtain that

$$\mathbf{E}_{s,0,1}[\, e^{-\alpha t}\, V^*(S_t, Y_t, \eta_t)\,] \,\leq\, \mathbf{E}_{s,0,1}[\, e^{-\alpha(t\wedge\tau_{s_0})}\, V^*(S_{t\wedge\tau_{s_0}}, 0, 1)\,]. \tag{3.14}$$

To prove (3.12), we want to apply Itô's formula on the above right-hand side followed by taking expectations.

Recall the operator L introduced on page 56. As the function $V^*(\cdot, 0, 1)$ defined to (s_0, ∞) can be extended to a C^2-function on $\mathbb{R}$, Itô's formula $\mathbf{P}_{s,0,1}$-a.s. yields

$$e^{-\alpha(t\wedge\tau_{s_0})}\, V^*(S_{t\wedge\tau_{s_0}}, 0, 1) \;=\; V^*(s, 0, 1) \,+ \int_0^{t\wedge\tau_{s_0}} e^{-\alpha u}(L - \alpha I)V^*(S_u, 0, 1)\, \mathrm{d}u \,+\, I_{BM}$$

where I_{BM} is an integrable stochastic integral against Brownian motion whose expectation vanishes.

Furthermore, by explicit calculation, $(L - \alpha I)V^*(s', 0, 1) = 0$ for $s' > s_0$, and hence the right-hand side of (3.14) reduces to $V^*(s, 0, 1)$ eventually showing (3.12).

It remains to verify (v3) for any $s > s_0$ in both cases (a) and (b).

For (a), observe that

$$V^*(s, 0, 1) \;=\; V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} \;\geq\; (K - b_0)\left(\frac{s_0}{b_0}\right)^{\gamma^-}\left(\frac{s}{s_0}\right)^{\gamma^-} \;=\; (K - b_0)\left(\frac{s}{b_0}\right)^{\gamma^-},$$

and hence it suffices to show that $(K - b_0)(s/b_0)^{\gamma^-} \geq (K - s_0)^+$ for $s > s_0$. Note that $(K - b_0)(\cdot/b_0)^{\gamma^-}$ coincides on $[b_0, \infty)$ with the value function $\tilde{V}$ given in Theorem 3.4 which satisfies both $\tilde{V}(s) \geq (K - s)^+$ for $s \in [b_0, \infty)$ and $\tilde{V}'(b_0) = -1$. Therefore, because $(K - b_0)(\cdot/b_0)^{\gamma^-}$ is a convex function on $(0, \infty)$, it must be bounded below by $(K - \cdot)^+$ on (s_0, b_0), too.

For (b), there is nothing to show if $s_0 \geq K$. But, if $b_0 \leq s_0 < K$, then we know from the proof of Lemma 3.8 that $(K - s_0)(s/s_0)^{\gamma^-} > (K - s)^+$ for all $s > s_0$. Thus

$$V^*(s, 0, 1) \;=\; V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} \;>\; (K - s)^+ \quad \text{for} \quad s > s_0$$

because $V(s_0, 1, 1) > (K - s_0)^+$ by assumption in case (b).

3.2.4 Proof of Theorem 3.6 (iii)(c)

We are going to verify (v1),(v2),(v3) for fixed $s > s_0$ as in Section 3.2.3 but using $\tau^* = \tau_{s_0}$.

First, note that, on $\{\tau_{s_0} < \infty\}$, the process $(S_t, 0 \le t \le \tau_{s_0})$ under $\mathbf{P}_{s,0,1}$ has the same distribution as the process $(\tilde{S}_t, 0 \le t \le \tilde{\tau}_{s_0})$ under $\tilde{\mathbf{P}}_s$ introduced in Theorem 3.4. By Lemma 3.8, for $s > s_0$, we therefore have

$$\begin{aligned}&\mathbf{E}_{s,0,1}[e^{-\alpha\tau_{s_0}}(K - S_{\tau_{s_0}})^+ \mathbf{1}_{\{\tau_{s_0}<\infty\}}]\\ &= \tilde{\mathbf{E}}_s[e^{-\alpha\tilde{\tau}_{s_0}}(K - \tilde{S}_{\tilde{\tau}_{s_0}})^+ \mathbf{1}_{\{\tilde{\tau}_{s_0}<\infty\}}] \;=\; (K - s_0)\left(\frac{s}{s_0}\right)^{\gamma^-}\end{aligned}$$

where $V(s_0, 1, 1) = K - s_0$ by assumption showing (v1).

For (v2), one can copy the corresponding proof in Section 3.2.3 because the value function has the same form in all sub-cases (a,b,c).

Finally, $V^*(s, 0, 1) = \tilde{V}_0(s) \ge (K - s)^+$ for $s > s_0$, showing (v3) and finishing the proof of Theorem 3.6(iii).

3.2.5 Proof of Theorem 3.6 (iv)

As in the proof of Theorem 3.6(ii), we first guess the structure of the stopping region and then verify that the solution of the corresponding free-boundary value problem is the wanted value function.

First, as the value function is claimed to be attained at the Markov time $\tau_{[b_*,b_0],0} \wedge \tau_{b_1} \wedge \tau_{\eta,0}$, the stopping region to be guessed should take the form

$$\Big[\, [b_*, b_0] \times \{0\} \times \{1\} \,\Big] \cup \Big[\, (0, b_1] \times \{1\} \times \{1\} \,\Big] \cup \Big[\, (0, \infty) \times \{1\} \times \{0\} \,\Big]$$

where b_0, b_1 are already known but $b_* \in [s_0, b_0)$ is not. Recall that both b_0 and b_1 must be less than K.

Now, referring to the proof of Theorem 3.6(ii) for the underlying argument, the corresponding free-boundary value problem for the unknown value function $V(s, 0, 1)$ is

$$0 \;=\; \mu_0 s\partial_1 V(s, 0, 1) + \frac{1}{2}\sigma_0^2 s^2 \partial_{11} V(s, 0, 1) - \alpha V(s, 0, 1) \qquad \text{for } s \in (s_0, b_*) \cup (b_0, \infty)$$

depending on the unknown b_* subject to the boundary conditions

$$\begin{aligned} V(s_0, 1, 1) &\;=\; V(s_0+, 0, 1);\\ V(b_*-, 0, 1) &\;=\; K - b_*;\\ \partial_1 V(b_*-, 0, 1) &\;=\; -1;\\ K - b_0 &\;=\; V(b_0+, 0, 1);\\ -1 &\;=\; \partial_1 V(b_0+, 0, 1);\\ \lim_{s\to\infty} V(s, 0, 1) &\;=\; 0. \end{aligned} \tag{3.15}$$

Taking into account Theorem 3.4, if $V(\cdot,0,1)$ satisfies these constraints then it must have the representation

$$\begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : \quad s \in (s_0, b_*), \\ K - s & : \quad s \in [b_*, b_0] \cap (s_0, \infty), \\ (K - b_0)\left(\dfrac{s}{b_0}\right)^{\gamma^-} & : \quad s \in (b_0, \infty), \end{cases} \tag{3.16}$$

where e_1^*, e_2^*, b_* should be determined by the first three conditions of (3.15). However, since V is the value function associated with an optimal stopping problem, there is a fourth constraint on the choice of e_1^*, e_2^*, b_* which is used in the next lemma.

Lemma 3.10. *If $s_0 < b_0$ and $(K - s_0) < V(s_0, 1, 1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, then there exist unique coefficients e_1^*, e_2^* and a unique stopping level $b_* \in (s_0, b_0)$ such that*

$$V(s_0, 1, 1) \;=\; e_1^* s_0^{\gamma+} + \; e_2^* s_0^{\gamma^-}; \tag{3.17}$$

$$e_1^* b_*^{\gamma^+} + \; e_2^* b_*^{\gamma^-} \;=\; K - b_*; \tag{3.18}$$

$$e_1^* \gamma^+ b_*^{\gamma^+} + \; e_2^* \gamma^- b_*^{\gamma^-} \;=\; -b_*;$$

$$e_1^* s^{\gamma^+} + \; e_2^* s^{\gamma^-} \;>\; K - s \quad \textit{for} \quad s \in (s_0, b_*).$$

Before proving this lemma, we are going to state the following preparatory results.

Lemma 3.11. *(see [44, Lemma 2] for example) For fixed $s_0 \le s \le \tilde{s}$,*

$$\phi_1(s, \tilde{s}) \stackrel{def}{=} \mathbf{E}_{s,0,1}[e^{-\alpha(\tau_{s_0} \wedge \tau^{\tilde{s}})} \mathbf{1}_{\{\tau_{s_0} < \tau^{\tilde{s}}\}}] = \frac{s^{\gamma^+} \tilde{s}^{\gamma^-} - s^{\gamma^-} \tilde{s}^{\gamma^+}}{s_0^{\gamma^+} \tilde{s}^{\gamma^-} - s_0^{\gamma^-} \tilde{s}^{\gamma^+}}$$

and

$$\phi_2(s, \tilde{s}) \stackrel{def}{=} \mathbf{E}_{s,0,1}[e^{-\alpha(\tau_{s_0} \wedge \tau^{\tilde{s}})} \mathbf{1}_{\{\tau_{s_0} > \tau^{\tilde{s}}\}}] = \frac{s_0^{\gamma^+} s^{\gamma^-} - s_0^{\gamma^-} s^{\gamma^+}}{s_0^{\gamma^+} \tilde{s}^{\gamma^-} - s_0^{\gamma^-} \tilde{s}^{\gamma^+}}$$

where

$$\tau^{\tilde{s}} \stackrel{def}{=} \inf\{t \ge 0 : S_t \ge \tilde{s}\}.$$

Lemma 3.12. *Under the assumptions of Lemma 3.10, the equation*

$$V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} \;=\; K - s$$

has exactly two solutions $s_1, s_2 \in (s_0, K)$ only one of which, say s_1, is less than b_0.

Proof. Consider the function $f(\cdot) = V(s_0, 1, 1)(\cdot/s_0)^{\gamma^-} - (K - \cdot)$ on $(0, \infty)$, and note that $f(s_0) > 0$, $f(b_0) < 0$, $f(K) > 0$. By the Intermediate Value Theorem, there exist $s_0 < s_1 < b_0$ and $b_0 < s_2 < K$ such that $f(s_1) = f(s_2) = 0$, that is,

$$V(s_0, 1, 1)\left(\frac{s_i}{s_0}\right)^{\gamma^-} = K - s_i \quad \text{for } i = 1, 2.$$

There exist exactly two points, only, because $V(s_0, 1, 1)(\cdot/s_0)^{\gamma^-}$ is strictly convex on $(0, \infty)$ and its graph can be intersected by a line at no more than two points. Moreover,

$$V(s_0, 1, 1)\left(\frac{s}{s_0}\right)^{\gamma^-} < K - s \quad \text{for} \quad s_1 < s < s_2, \tag{3.19}$$

because of strict convexity, too. □

Proof of **Lemma 3.10**. Introduce

$$\Gamma(s, \tilde{s}) \stackrel{\text{def}}{=} V(s_0, 1, 1)\phi_1(s, \tilde{s}) + (K - \tilde{s})\phi_2(s, \tilde{s}) \quad \text{for} \quad s_0 \leq s \leq \tilde{s}$$

using the functions ϕ_1, ϕ_2 defined in Lemma 3.11. For fixed $\tilde{s} > s_0$, note that the function $[s_0, \tilde{s}\,] \ni s \mapsto \Gamma(s, \tilde{s})$ is of the form $e_1\, s^{\gamma^+} + e_2\, s^{\gamma^-}$ with

$$e_1 = \frac{V(s_0, 1, 1)\tilde{s}^{\,\gamma^-} - (K - \tilde{s})s_0^{\gamma^-}}{s_0^{\gamma^+}\tilde{s}^{\,\gamma^-} - s_0^{\gamma^-}\tilde{s}^{\,\gamma^+}}, \quad e_2 = \frac{-V(s_0, 1, 1)\tilde{s}^{\,\gamma^+} + (K - \tilde{s})s_0^{\gamma^+}}{s_0^{\gamma^+}\tilde{s}^{\,\gamma^-} - s_0^{\gamma^-}\tilde{s}^{\,\gamma^+}},$$

and that both boundary conditions

$$e_1\, s_0^{\gamma^+} + e_2\, s_0^{\gamma^-} = \Gamma(s_0, \tilde{s}) = V(s_0, 1, 1), \quad e_1\, \tilde{s}^{\,\gamma^+} + e_2\, \tilde{s}^{\,\gamma^-} = \Gamma(\tilde{s}, \tilde{s}) = K - \tilde{s}$$

are satisfied. Hence, if we can show that there is exactly one $b_* \in (s_0, b_0)$ such that both

$$\partial_1\Gamma(b_*, b_*) = -1 \quad \text{and} \quad \Gamma(s, b_*) > K - s \text{ for } s \in (s_0, b_*),$$

then the triplet (e_1^*, e_2^*, b_*), where e_1^*, e_2^* are given by the above formulae for e_1, e_2 when replacing $\tilde{s}$ by b_*, would be the unique solution of the problem stated in Lemma 3.10. Here the uniqueness of e_1^*, e_2^* follows from the uniqueness of b_* as the formulae for e_1^*, e_2^* coincide with the unique solution to the sub-system (3.17),(3.18) of the conditions in Lemma 3.10 when treating $s_0^{\gamma^\pm}$ and $b_*^{\gamma^\pm}$ as coefficients.

First, we show that that there is $b_* \in (s_0, b_0)$ such that $\partial_1\Gamma(b_*, b_*) = -1$. For the uniqueness of b_* we refer to Remark 3.13(ii) below.

Using simple calculations based on Itô's formula, observe that, for any $s \geq s_0$, the stochastic process $(e^{-\alpha(t\wedge\tau_{s_0})}V(s_0,1,1)(S_{t\wedge\tau_{s_0}}/s_0)^{\gamma^-},\ t\geq 0)$ is a $\mathbf{P}_{s,0,1}$-martingale. Now recall s_1 from Lemma 3.12 and the stochastic representation of $\Gamma(s,\tilde{s})$ in terms of ϕ_1,ϕ_2. Then, by Doob's Optional Sampling Theorem, for $s_0\leq s\leq s_1$,

$$\begin{aligned}
V(s_0,1,1)\left(\frac{s}{s_0}\right)^{\gamma^-} &= \mathbf{E}_{s,0,1}[\, e^{-\alpha(\tau_{s_0}\wedge\tau^{s_1})}V(s_0,1,1)\left(\frac{S_{\tau_{s_0}\wedge\tau^{s_1}}}{s_0}\right)^{\gamma^-}] \\
&= V(s_0,1,1)\phi_1(s,s_1)+V(s_0,1,1)\left(\frac{s_1}{s_0}\right)^{\gamma^-}\phi_2(s,s_1) \\
&= V(s_0,1,1)\phi_1(s,s_1)+(K-s_1)\phi_2(s,s_1) \\
&= \Gamma(s,s_1)
\end{aligned}$$

using Lemma 3.12 to justify the penultimate equality above. Thus

$$\partial_1\Gamma(s_1,s_1) \;=\; \partial_s\left[V(s_0,1,1)\left(\frac{s}{s_0}\right)^{\gamma^-}\right]_{s=s_1}. \tag{3.20}$$

Next, for any $s>s_0$,

$$\partial_1\Gamma(s,s) \;=\; \frac{V\, s^{\gamma^++\gamma^--1}\,(\gamma^+-\gamma^-)+(K-s)\,s^{-1}\,(\gamma^- s_0^{\gamma^+}\,s^{\gamma^-}-\gamma^+ s_0^{\gamma^-}\,s^{\gamma^+})}{s_0^{\gamma^+}\,s^{\gamma^-}\;-\;s_0^{\gamma^-}\,s^{\gamma^+}}$$

where V stands for $V(s_0,1,1)$. Consider the above right-hand side as a function of (V,s) which we denote by $g(V,s)$ in what follows.

Choose $s=b_0$ and realise that the function $V\mapsto g(V,b_0)$ is strictly decreasing since $s_0<b_0$ implies $s_0^{\gamma^+}{b_0}^{\gamma^-}\;-\;s_0^{\gamma^-}{b_0}^{\gamma^+}<0$. Moreover $g((K-b_0)(s_0/b_0)^{\gamma^-},b_0)=-1$, so that our assumption of $V(s_0,1,1)<(K-b_0)(s_0/b_0)^{\gamma^-}$ implies $g(V(s_0,1,1),b_0)>-1$.

But, using (3.20), we also have that

$$g(V(s_0,1,1),s_1) \;=\; \partial_s\left[V(s_0,1,1)\left(\frac{s}{s_0}\right)^{\gamma^-}\right]_{s=s_1},$$

where, by the same arguments used in the proof of Lemma 3.12, the above right-hand side must be less than -1.

All in all, we obtain that $g(V(s_0,1,1),s_1)\;<\;-1\;<\;g(V(s_0,1,1),b_0)$. And since the function $g(V(s_0,1,1),\cdot)$ is continuous on (s_0,∞), and since $s_0<s_1<b_0$ by Lemma 3.12, it is again a consequence of the Intermediate Value Theorem that there exists $b_*\in(s_1,b_0)$ such that $\partial_1\Gamma(b_*,b_*)=g(V(s_0,1,1),b_*)=-1$.

Second, finishing the proof of Lemma 3.10, we will show that

$$\Gamma(s, b_*) \;=\; e_1^* s^{\gamma^+} \,+\, e_2^* s^{\gamma^-} \;>\; K - s \quad \text{for } s \in (s_0, b_*)$$

where

$$e_1^* \;=\; \frac{V(s_0,1,1) b_*^{\gamma^-} - (K - b_*) s_0^{\gamma^-}}{s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}}, \quad e_2^* \;=\; \frac{-V(s_0,1,1) b_*^{\gamma^+} + (K - b_*) s_0^{\gamma^+}}{s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}}.$$

In order to do so we analyse the function $f(s) = e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-}$, $s > 0$. Since $f'(b_*) = -1$ has already been shown, $f(s) > K - s$ for $s \in (s_0, b_*)$ would follow from f being strictly convex on $(0, b_*)$ which we are going to prove below.

First observe that both e_1^* and e_2^* are positive. In fact, as $s_0 < b_*$ implies that the denominator $s_0^{\gamma^+} b_*^{\gamma^-} - s_0^{\gamma^-} b_*^{\gamma^+}$ is negative, the positivity of the two coefficients e_1^*, e_2^* follows from

$$V(s_0,1,1) b_*^{\gamma^-} - (K - b_*) s_0^{\gamma^-} < 0 \quad \text{and} \quad - V(s_0,1,1) b_*^{\gamma^+} + (K - b_*) s_0^{\gamma^+} < 0$$

where the former inequality is a consequence of (3.19) because b_* was chosen from the interval (s_1, b_0) while the latter inequality is a consequence of our assumption $K - s_0 < V(s_0,1,1)$, on the one hand hand, and $s_0 < b_*$, $\gamma^+ > 0$, on the other.

As a consequence, if $\gamma^+ \geq 1$, then f is the sum of two strictly convex functions, and hence strictly convex everywhere.

Now assume $0 < \gamma^+ < 1$ which is the remaining case. Note that f has exactly one local minimum at $s' = [\frac{-e_2^* \gamma^-}{e_1^* \gamma^+}]^{\frac{1}{\gamma^+ - \gamma^-}}$ and that this minimum is global because $f(0+) = \lim_{s \to \infty} f(s) = \infty$.

Thus, f is strictly decreasing on $(0, s')$ and strictly increasing on (s', ∞) which implies $b_* < s'$ because $f'(b_*) = -1$.

Finally, f has exactly one point of inflection at $s'' = [\frac{-e_2^* \gamma^- (\gamma^- - 1)}{e_1^* \gamma^+ (\gamma^+ - 1)}]^{\frac{1}{\gamma^+ - \gamma^-}}$, and this point satisfies $s'' > s'$. Hence, f must be at least strictly convex on $(0, s')$ which also proves its strict convexity on $(0, b_*) \subseteq (0, s')$. □

Remark 3.13. (i) In the case of $K - s_0 = V(s_0,1,1)$, the choice of b_* has not been discussed yet. In this case, we claim that $b_* = s_0$, and we will verify below that the corresponding function given by (3.16) coincides with $V(\cdot, 0, 1)$ on (s_0, ∞).

(ii) In the case of $(K - s_0) < V(s_0,1,1) < (K - b_0)(s_0/b_0)^{\gamma^-}$, we showed existence of $b_* \in (s_0, b_0)$, and any such b_* uniquely determines a function as given by (3.16). We will verify below that any such function coincides with $V(\cdot, 0, 1)$ on (s_0, ∞). Moreover, as shown in the proof of Lemma 3.10, for any choice of b_*, the corresponding function given by (3.16) must be strictly convex on (s_0, b_*). Hence, as the value function with respect to an optimal stopping problem is unique, there can only be one b_*.

Set $\tau^* = \tau_{[b_*,b_0],0} \wedge \tau_{b_1} \wedge \tau_{\eta,0}$ and introduce

$$V^*(s,y,i) \stackrel{\text{def}}{=} \begin{cases} e_1^* s^{\gamma^+} + e_2^* s^{\gamma^-} & : \ s \in (s_0, b_*), y = 0, i = 1; \\ K - s & : \ s \in [b_*, b_0] \cap (s_0, \infty), y = 0, i = 1; \\ (K - b_0)\left(\dfrac{s}{b_0}\right)^{\gamma^-} & : \ s \in (b_0, \infty), y = 0, i = 1; \\ V(s,1,i) & : \ s > 0, y = 1, i \in \{0,1\}. \end{cases}$$

Again, by verifying the conditions (v1),(v2),(v3) stated at the beginning of Section 3.2.3 for any fixed $s > s_0$, we complete both the program set out in Remark 3.13 and the proof of Theorem 3.6(iv).

For (v1), if $s \in [b_*, b_0]$, then there is nothing to prove as one stops immediately, and if $s > b_0$, then (v1) follows from Theorem 3.4 as $V^*(\cdot, 0, 1)$ coincides with $\tilde{V}$ on (b_0, ∞).

In the remaining case of $s_0 < s < b_*$, observe that the assumptions of Lemma 3.10 are satisfied as $K - s_0 = V(s_0, 1, 1)$ can be ruled out. Furthermore,

$$\begin{aligned} &\mathbf{E}_{s,0,1}[e^{-\alpha\tau_{[b_*,b_0],0}\wedge\tau_{b_1}\wedge\tau_{\eta,0}}(K - S_{\tau_{[b_*,b_0],0}\wedge\tau_{b_1}\wedge\tau_{\eta,0}})^+] \\ =&\mathbf{E}_{s,0,1}[e^{-\alpha(\tau_{b_1}\wedge\tau_{\eta,0})}(K - S_{\tau_{b_1}\wedge\tau_{\eta,0}})^+\mathbf{1}_{\{\tau_{[b_*,b_0],0} > \tau_{s_0}\}}] + \mathbf{E}_{s,0,1}[e^{-\alpha\tau_{b_*}}(K - b_*)^+\mathbf{1}_{\{\tau_{[b_*,b_0],0} < \tau_{s_0}\}}] \\ =&\mathbf{E}_{s,0,1}\Big[\ \mathbf{E}_{s,0,1}[\, e^{-\alpha(\tau_{b_1}\wedge\tau_{\eta,0})}(K - S_{\tau_{b_1}\wedge\tau_{\eta,0}})^+\mathbf{1}_{\{\tau_{[b_*,b_0],0} > \tau_{s_0}\}}|\mathcal{F}_{\tau_{s_0}}\,]\ \Big] + (K - b_*)\,\phi_2(s, b_*) \end{aligned}$$

which, by strong Markov property, simplifies to

$$V(s_0, 1, 1)\,\phi_1(s, b_*) + (K - b_*)\,\phi_2(s, b_*).$$

Using the definition of Γ given at the beginning of the proof of Lemma 3.10 on page 68, the last expression equals $\Gamma(s, b_*)$. And since b_* was chosen such that $\Gamma(s, b_*) = V^*(s, 0, 1)$, condition (v1) follows in the remaining case, too.

For (v2), realise that, by the same arguments used in Section 3.2.3 on page 66, the above candidate value function of this section satisfies (3.14), and hence we only need to show that

$$\mathbf{E}_{s,0,1}[\, e^{-\alpha(t\wedge\tau_{s_0})}\, V^*(S_{t\wedge\tau_{s_0}}, 0, 1)\,] \ \le\ V^*(s, 0, 1) \tag{3.21}$$

for any $t \ge 0$.

Now, consider the function $\phi(s') = V^*(s', 0, 1)$, $s' > s_0$. Note that ϕ can be extended to a function on $(0, \infty)$ which is of the type described in Remark 3.9 having finitely many exceptional points of insufficient smoothness. In the case of $K - s_0 = V(s_0, 1, 1)$, where

$b_* = s_0$, there is one exceptional point at b_0, whereas in the case of $K - s_0 < V(s_0, 1, 1)$ there are two exceptional points at b_*, b_0. However, in both cases, applying Theorem 3.4 and Lemma 3.10, respectively, one can extend ϕ in such a way that ϕ' is continuous.

Thus, according to Remark 3.9, we $\mathbf{P}_{s,0,1}$-a.s. have

$$e^{-\alpha(t\wedge\tau_{s_0})}V^*(S_{t\wedge\tau_{s_0}}, 0, 1) \;=\; V^*(s, 0, 1) + \int_0^{t\wedge\tau_{s_0}} e^{-\alpha u}(L - \alpha I)V^*(S_u, 0, 1)\mathrm{d}u + I_{BM} \quad (3.22)$$

where L stands once more for the operator introduced on page 56, and I_{BM} is an integrable stochastic integral against Brownian motion whose expectation vanishes.

Next, by explicit calculation, $(L-\alpha I)V^*(\cdot, 0, 1) \leq 0$ on $(s_0, b_*) \cup (b_*, b_0) \cup (b_0, \infty)$. Since the process $(S_u,\, u \in [0, \tau_{s_0}))$ has $\mathbf{P}_{s,0,1}$ - a.s. no occupation time in $\{b_*, b_0\}$, inequality (3.21) follows from (3.22) by taking expectations, proving (v2).

Finally, condition (v3) is a consequence of Lemma 3.10, if $s \in (s_0, b_*)$, and of Theorem 3.4, if $s \in (b_0, \infty)$. Otherwise, there is nothing to be shown.

3.3 Numerical Analysis and Discussion

We are going to discuss the four cases (iii)(a-c) and (iv) of Theorem 3.6 using practically relevant values for $s_0, \mu_0, \sigma_0, \mu_1,\, \sigma_1, \alpha, \lambda, K$.

Note that the choice of $\mu_1, \sigma_1, \lambda, K$ fixes the value of b_1 and that the two cases (iii)(b,c) of Theorem 3.6 can be reformulated as

(b') $b_0 \leq s_0$ and $b_1 < s_0$;

(c') $b_0 \leq s_0 < K$ and $b_1 \geq s_0$.

However, the formulation of the two cases (iii)(a) and (iv) requires the value of $V(s_0, 1, 1)$, and that's why we decided to formulate (iii)(b,c) using $V(s_0, 1, 1)$, too.

In what follows, when using the noun 'put' without further specification, we mean a *perpetual* American put as considered in Theorem 3.6. However, as motivated in Remark 3.1(ii,iii,v), the average length of the put's optimal exercise time is supposed to be rather short, and hence we think that our analysis also produces good benchmarks for traded American puts with times to maturity being long enough to allow for medium term option trading, that is, three months and longer.

First, we have to choose the put's underlying asset. By the macroeconomic explanation given by Black in [9], we think that a leverage effect is more likely to be observed when the whole market falls, and hence we choose an index, say, the Dow Jones index.

Second, to fully determine the put, a strike level has to be chosen. At the end of this section we give a summary of how to choose the strike level motivated by our discussion of Theorem 3.6 below. For now we choose $K = 17000$ for demonstration.

Next we fix the following hypothetical values for $\sigma_0 = 20\%$, $\mu_1 = 0$, $\sigma_1 = 35\%$, $\alpha = 5\%$, $\lambda = 100$ and refer to Remark 3.1 and Remark 3.3(i) for their interpretation.

So, $\sigma_0 = 20\%$ is supposed to be the implied volatility at present time of a traded American put with strike K and time to maturity of at least three months, and we assume that the expected market drop would cause an 'excited' volatility of $\sigma_1 = 35\%$.

Setting $\lambda = 100$ means to assume that the 'excited' state would only last for half a week on average.

After the market has dropped, it is not clear what the new trend μ_1 of the index would be. Furthermore, if the 'excited' state only lasts for a short period of time, one can assume that the 'excited' fluctuations according to the bigger σ_1 dominate the trend. Thus a reasonable choice for the new trend would be $\mu_1 = 0$.

The two remaining non-fixed parameters are s_0 and μ_0. Since σ_0, α, K have been fixed, there is a one-to-one correspondence between μ_0 and b_0, and hence, each pair (μ_0, s_0) determines one of the four cases (iii)(a-c) and (iv) of Theorem 3.6. The optimal stopping rules given in each of these cases are called *strategies* of the trader, in what follows.

In practice, depending on the present value s of the Dow Jones, the trader would choose s_0 according to their preferences of the future—they expect a market drop of a certain size. In our analysis we take the reverse point of view: we first classify the values of s_0 and then discuss the impact of present values s above s_0 on the strategy to be chosen by the trader.

While $s - s_0$ determines the size of the expected market drop, the choice of μ_0 determines how soon this is supposed to happen in terms of the model (recall that σ_0 has been fixed). A rather small value of μ_0 should be used if one wants that many price-trajectories predicted by the model reach the level s_0 in a rather short time. For example, a value of $\mu_0 = -100\%$ would imply that, roughly, the value of the index expected under the model drops from 15600 to 15000 within two weeks.

In contrast, in the case of bigger values of μ_0, the model more often predicts rising values of the index in the future, and this is of course not in accordance with an expected market drop. We will nevertheless analyse bigger values of μ_0 because the corresponding strategies might be of use for the trader in case they learn during the trade that their preferences of the future were wrong.

Figure 1 below shows the blue graph of $b_0 = b_0(\mu_0)$ embedded into the (μ_0, s_0) - plane. The red horizontal line marks the level $b_1 = 14658$ which crosses $b_0(\mu_0)$ at $\mu_0 = 13.7\%$.

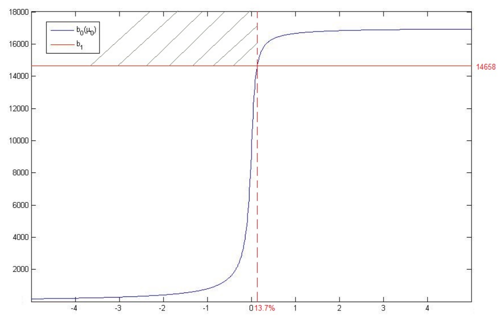


Figure 1

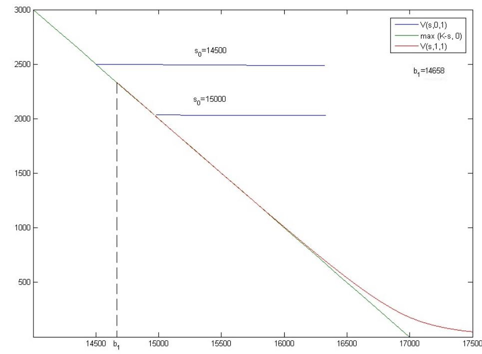


Figure 2

Any point (μ_0, s_0) left (or above) of the curve $b_0(\mu_0)$ is associated with one of the cases (iii)(b,c), and any point right of (or below) the curve is associated with one of the cases (iii)(a),(iv) of Theorem 3.6.

Figure 2 shows the value functions corresponding to the two points $(-1, 14500)$ and $(-1, 15000)$ which are both left of the curve $b_0(\mu_0)$ in Figure 1. The green anti-diagonal line is part of the gain function $(K-\cdot)^+$, and the red convex curve is the graph of $V(\cdot, 1, 1)$ which merges onto the gain function at b_1. The blue branches hitting $V(\cdot, 1, 1)$ at $s_0 = 14500$ and $s_0 = 15000$, respectively, are the graphs of the corresponding version of $V(\cdot, 0, 1)$ before the regime change at s_0.

Recall that $V(\cdot, 0, 1)$ and $V(\cdot, 1, 1)$ are two different components of the value function, and they only meet continuously in the above picture because of the boundary condition explained in Remark 3.2(i) on page 56.

Figure 2 can be used to illustrate the qualitative difference between the strategy assuming $s_0 = 14500 \leq b_1$ (case (iii)(c)) and the one assuming $s_0 = 15000 > b_1$ (case (iii)(b)). When assuming $s_0 = 14500$, the trader waits for the index to reach s_0 and would then sell/exercise the put immediately. When assuming $s_0 = 15000$, they would also wait for the index to reach s_0 but would then exploit the regime change from σ_0 to σ_1 implemented into their model due to an implied leverage effect during a market fall: they would either sell/exercise the put after a further waiting time of the order of $1/\lambda$, or sell/exercise the put when the index reaches b_1. Note that, in our example, $1/\lambda$ equals 1/2 week which is very short. As a consequence, $V(\cdot, 1, 1)$ looks very similar to how the value function of a traded American put shortly before maturity would look like, and this explains why $V(\cdot, 1, 1)$ is so close to the gain function.

Remark 3.14. (i) According to our definition of the value function, all strategies refer to exercising the option. However, since the price of an option which has not matured yet always tops its exercise value, selling the option would not cause any disadvantage.

(ii) As argued above, choosing a put with strike K such that the level s_0 defining the trade is below b_1 and then applying Theorem 3.6 using a small value of μ_0, that is, using a value of μ_0 in accordance with an expected market drop results in an optimal strategy where the trader would NOT benefit from the implied leverage effect. So, the trader would want to choose K such that the level s_0 they have in mind is above b_1.

(iii) Following (ii), the strategy to be used would be the one described above with respect to $s_0 = 15000$ (case (iii)(b)) for at least all (μ_0, s_0) in the marked area shown in Figure 1. Notice that this area covers values of μ_0 as large as 13.7% which is, for example, well-above the 10 years (2004-2013) average return rate of 6.05% of the Dow Jones. Thus, the strategy given in case (iii)(b) of Theorem 3.6 is robust in the sense that it applies to small values of μ_0, when the model would predict a market drop in accordance with the preferences of the trader, but also to 'neutral' values of μ_0, when the model would predict standard returns rather than a market drop.

Because of Remark 3.14(ii), we restrict the remaining part of our discussion to cases where $s_0 > b_1$. By Remark 3.14(iii), we know that the strategy given in case (iii)(b) is robust for small and 'neutral' values of μ_0. Next, we discuss the type of strategy offered by Theorem 3.6 when the trader's preferences for the future are 'entirely' wrong, that is, when μ_0 is significantly bigger than 13.7% and (μ_0, s_0) belongs to the quadrant on the right-hand side of the marked area in Figure 1. For demonstration, we choose $\mu_0 = 30\%$.

Figure 3 shows the part of the quadrant on the right-hand side of the marked area in Figure 1 which refers to $10\% \leq \mu_0 \leq 100\%$. The blue upper concave curve is the graph of $b_0(\mu_0)$, and the green concave curve beneath, which meets the upper curve at $\mu_0 = 13.7\%$, is the graph of a function we call $s_0^{max} = s_0^{max}(\mu_0)$. This function gives the root of the equation

$$V(s_0, 1, 1) \;=\; (K - b_0)\left(\frac{s_0}{b_0}\right)^{\gamma^-}, \quad s_0 \text{ unknown},$$

which is the value of s_0 at which the switch between case (iii)(a) and case (iv) of Theorem 3.6 occurs. The red horizontal line again marks the level of $b_1 = 14658$, and the black vertical fat bar marks the values of s_0 between b_1 and $s_0^{max} = 15742$ at $\mu_0 = 30\%$.

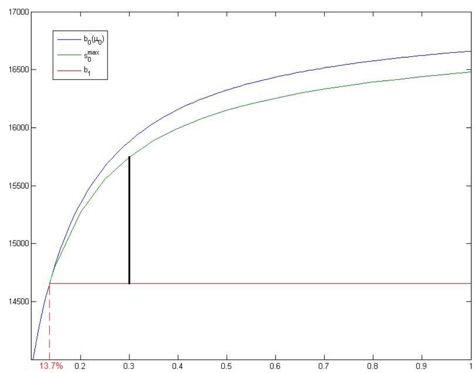


Figure 3

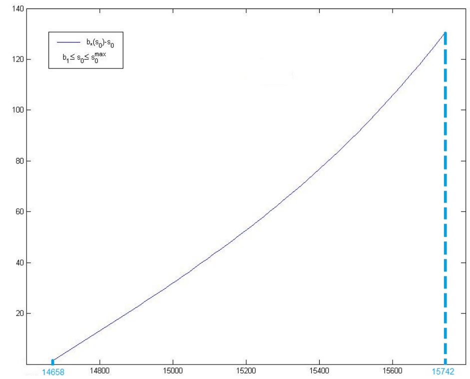


Figure 4

Any point (μ_0, s_0) between the horizontal line and the curve $s_0^{max}(\mu_0)$ is associated with case (iv) of Theorem 3.6, while any point between the two curves $s_0^{max}(\mu_0)$ and $b_0(\mu_0)$ is associated with case (iii)(a).

In case (iv), there exists a corresponding $b_* = b_*(\mu_0, s_0) \in (s_0, b_0)$. For fixed $\mu_0 = 30\%$, we write $b_*(s_0)$ for $b_*(0.3\,, s_0)$, and Figure 4 shows the graph of $b_*(s_0) - s_0$ for those values of s_0 marked by the vertical fat bar in Figure 3. Note that a further look at the proof of Lemma 3.10 reveals $\lim_{s_0 \uparrow s_0^{max}} b_*(s_0) = b_0$.

Figure 5 below shows the value function corresponding to the point $(0.3\,, 15000)$ which is a point on the vertical fat bar in Figure 3. To better illustrate the typical shape of the components of this value function, we scaled the axes in a non-linear way which is why, in contrast to the other figures, there are no numerical values assigned to the axes.

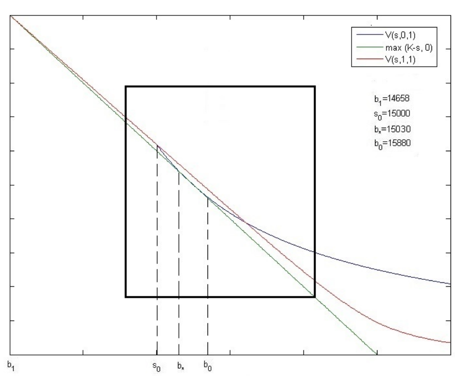


Figure 5

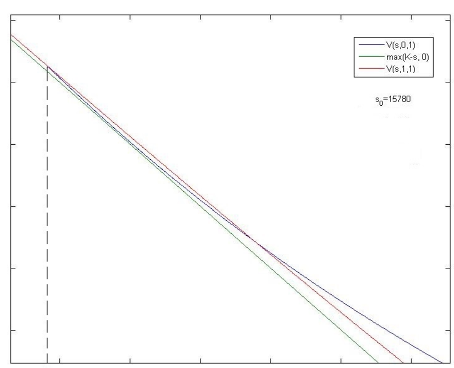


Figure 6

The green anti-diagonal line is part of the gain function $(K - \cdot)^+$, and the red convex curve is the graph of $V(\cdot, 1, 1)$ which merges onto the gain function at b_1 in the upper left corner. The component $V(\cdot, 0, 1)$, plotted in blue, is identical to the gain function between b_* and b_0 but also has two branches: the left branch below $V(\cdot, 1, 1)$ connects $V(\cdot, 1, 1)$ at s_0 with the gain function at b_*, and the right branch crosses $V(\cdot, 1, 1)$ before merging onto the gain function at b_0.

Figure 6 zooms into the window marked in Figure 5 showing the components of the value function corresponding to the point $(0.3\,,15780)$ which is a case-(iii)(a)-point above the vertical fat bar in Figure 3 but still below the curve $b_0(\mu_0)$. Only the component $V(\cdot,0,1)$ changes. While, in Figure 5, $V(\cdot,0,1)$ is identical to the gain function on a whole interval (b_*,b_0), in Figure 6, its graph stays above the gain function everywhere crossing $V(\cdot,1,1)$ from the right and meeting it again further left at s_0.

Figure 6 graphically confirms Theorem 3.6 in asserting that the case-(iii)(a)-strategy is identical to the case-(iii)(b)-strategy discussed in the context of Figure 2. All in all, the case-(iii)(b)-strategy would be applicable for all (μ_0, s_0) in both the marked area shown in Figure 1 and in that part of the quadrant on the right-hand side of this marked area which is above the curve $s_0^{max}(\mu_0)$ in Figure 3.

Remark 3.15. Recall that the case-(iii)(b)-strategy involves waiting for the index to fall $s-s_0$ points where, by Remark 3.1(ii), the size of $s-s_0$ is considerable. Thus, for values of μ_0 as large as 30% in our example, one would expect the index to take a rather long time for dropping as much as $s-s_0$. To avoid this risk, the trader would not want to choose a put with strike K such that the level s_0 defining the trade is above the curve $s_0^{max}(\mu_0)$ in Figure 3 for a range of 'larger' values of μ_0.

The alternative to this unsuitable choice of K would be to choose a put with strike K such that the level s_0 stays below the curve $s_0^{max}(\mu_0)$ in Figure 3 for all 'larger' values of μ_0. This alternative refers to the remaining case-(iv)-strategy, and we return to Figure 5 to discuss this strategy in more detail.

Recall that $\mu_0 = 30\%$ and $s_0 = 15000$ in our example. According to the function $b_*(s_0) - s_0$ shown in Figure 4, the gap between s_0 and b_* in our example is about 30 points of the Dow Jones index. Clearly, if $s - s_0$ is of the order of 50 points and the value s of the Dow Jones is of the order of 15000 points, then a drop from s to s_0 would not have any effect on the volatility of the index. So, at least in our example, to be in agreement with the model's assumptions, the value s should be well above b_* (i.e. $b_* \ll s$).

Figure 5 can now be used to illustrate the two different strategies depending on how much the present value s is above b_*. When assuming $b_* < s < b_0$, the trader would sell/exercise immediately, while, when assuming $b_0 < s$, the trader waits for the index to reach b_0 and would then sell/exercise. By the same reason given in Remark 3.15, the trader would not want to wait for the index to reach b_0 if μ_0 is as large as 30%. Therefore, a further but final constraint on where the present value s should be located is $b_* \ll s < b_0$. Note that $b_0 - b_*$ is of the order of 800 points in our example which is on the right scale for taking into account a possible leverage effect if the index drops from s satisfying $b_* \ll s < b_0$ to a level s_0 below $b_* = 15030$. For given s and s_0, the relation $b_*(\mu_0, s_0) < s < b_0(\mu_0)$ wanted for all 'larger' values of μ_0 can be achieved by choosing an appropriate strike level K.

3.4 Choosing the Strike Level

The following steps present a summary of the previous discussion on how to choose the strike level of the put depending on both the stopping levels b_0, b_1, b_* given by Theorem 3.6 and the level s_0^{max} introduced in the paragraph preceding Figure 3 above. After this summary we briefly describe how the strategies given in Theorem 3.6 could be used for trading.

Step 1: Fix a discount rate α and choose an index with present value s. Find σ_0 by comparing implied volatilities calculated from a range of traded options on the index. Decide about the size of $s - s_0$ the index is expected to drop in the near future. Based on analysing historical data or otherwise, decide about the size of the 'excited' volatility σ_1. Analysing historical data or otherwise, find the average time span of an 'excited' volatility regime after a drop of size $s - s_0$ of the index of your choice, that is, find $1/\lambda$. Set $\mu_1 = 0$.

Step 2: For different values of K calculate: b_1; $\tilde{\mu}_0$ such that $b_1 = b_0(\tilde{\mu}_0)$; $s_0^{max}(\tilde{\mu}_0 + \rho_0)$ for sufficiently large ρ_0; $b_0(\tilde{\mu}_0 + \rho_0)$; $b_*(\tilde{\mu}_0 + \rho_0, s_0)$. For the right tuning of ρ_0 compare with both Figure 3 where $\tilde{\mu}_0$ and $\tilde{\mu}_0 + \rho_0$ were 13.7% and 30%, respectively, and the comments in Remark 3.14(iii) about the magnitude of 13.7%.

Step 3: Finally choose a put with strike level K such that $b_1 < s_0 < s_0^{max}(\tilde{\mu}_0 + \rho_0)$ and $b_*(\tilde{\mu}_0 + \rho_0, s_0) < s < b_0(\tilde{\mu}_0 + \rho_0)$. We think that, for trading, the present value s of the index and the drop-to-level s_0 would be placed best if the size of $b_*(\tilde{\mu}_0 + \rho_0, s_0) - s_0$ is on a smaller scale than $s - s_0$ as in our example above. Note that this would also entail $b_*(\tilde{\mu}_0 + \rho_0, s_0) \ll s$.

When trading a put of the above choice using the strategies given by Theorem 3.6, assuming that the value of μ_0 is sufficiently negative to be conform with a drop of size $s - s_0$ in the near future, the trader would initially follow the case (iii)(b) strategy.

First, if the level s_0 is reached within the expected time frame, the trader would continue following the case (iii)(b) strategy to the end. In practical terms, the exponential waiting time should be realised by waiting a multiple of the average waiting time $1/\lambda$ where the choice of the multiple is up to the trader.

Second, if the level s_0 is <u>not</u> reached within the expected time frame, the trader would have gained enough new market data to update the value of μ_0. Based on statistical testing or otherwise, they should decide whether the updated value of μ_0 is below $\tilde{\mu}_0$ or above $\tilde{\mu}_0 + \rho_0$.

If the decision is for the updated μ_0 to be below $\tilde{\mu}_0$, the trader could continue following the case (iii)(b) strategy (updating μ_0 again if necessary), but they should also consider to finish the trade as soon as selling/exercising would not result in any losses.

If the decision is for the updated μ_0 to be above $\tilde{\mu}_0+\rho_0$, the trader should change to the case (iv) strategy but with respect to the most recent value of the underlying asset. If this new present value s is above $b_*(\tilde{\mu}_0+\rho_0, s_0)$, they should sell/exercise immediately. However, if it is in the range of s_0 to $b_*(\tilde{\mu}_0+\rho_0, s_0)$, the trader could continue following the primary case (iii)(b) strategy, unless the level $b_*(\tilde{\mu}_0+\rho_0, s_0)$ is reached before the drop-to-level s_0 when they should sell/exercise immediately.

3.5 Chapter Appendix

We verify that the explicit expression given for $V(\cdot, 1, 1)$ in Theorem 3.6(ii) is indeed the value function. This constitutes a degenerate case of the Regime Switching model. By the argument in Corollary 2.10 and Remark 2.11 (v), if we can show the solution to the free-boundary problem is unique, then we do not need to prove properties similar to (v2) and (v3) on page 63. We have tried but failed to find an approach to prove Theorem 3.6(ii) using a 'guess and verify' approach.

Theorem 3.6(ii) corresponds to the Regime Switching model with $\mu_2 = 0, \sigma_2 = 0$, $r_1 = r_2 = r$, with Q-matrix

$$\begin{pmatrix} -\lambda & \lambda \\ 0 & 0 \end{pmatrix}.$$

In this case, it is only necessary to find a single stopping level b_1. The optimal stopping level b_1 must satisfy (3.10) because $V(\cdot, 1, 1)$ satisfies (3.8). However, finding b_1 by solving (3.10) requires showing uniqueness of solutions to a non-linear equation.

Lemma 3.16. *There is exactly one solution $(c_1, c_2, d_2, b_1) \in \mathbb{R}^3 \times (0, K)$ to the system (3.10).*

Proof. We only show the lemma in the case of $\alpha+\lambda \neq \mu_1$ using the corresponding function h given in Remark 3.7(ii) because nothing fundamental changes when doing the calculation in the remaining single case of $\alpha+\lambda = \mu_1$ with another function h.

First, we ignore the last equation of the system (3.10) and replace b_1 by an arbitrary $b > 0$. The resulting system of equations reads

$$\begin{aligned}
c_1 K^{\beta^+} + c_2 K^{\beta^-} - \frac{\lambda K}{\alpha+\lambda-\mu_1} + \frac{\lambda K}{\alpha+\lambda} &= d_2 K^{\beta^-}, \\
c_1 \beta^+ K^{\beta^+} + c_2 \beta^- K^{\beta^-} - \frac{\lambda K}{\alpha+\lambda-\mu_1} &= d_2 \beta^- K^{\beta^-}, \\
K - b &= c_1 b^{\beta^+} + c_2 b^{\beta^-} - \frac{\lambda b}{\alpha+\lambda-\mu_1} + \frac{\lambda K}{\alpha+\lambda},
\end{aligned}$$

and, for each $b > 0$, this system admits a unique solution c_1, c_2, d_2.

To analyse the non-linear last equation of the system (3.10), we only need to know c_1 and c_2 which are explicitly given by

$$\begin{aligned} c_1 &= \frac{\lambda K^{1-\beta^+}}{(\beta^- - \beta^+)}\left[\frac{\beta^-}{\alpha+\lambda-\mu_1} - \frac{\beta^-}{\alpha+\lambda} - \frac{1}{\alpha+\lambda-\mu_1}\right], \\ c_2(b) &= \left[K - b + \frac{\lambda b}{\alpha+\lambda-\mu_1} - \frac{\lambda K}{\alpha+\lambda} - c_1 b^{\beta^+}\right]\frac{1}{b^{\beta^-}}. \end{aligned}$$

Second, for each $b > 0$, we introduce the function

$$V_b(s) = c_1 s^{\beta^+} + c_2(b) s^{\beta^-} - \frac{\lambda s}{\alpha+\lambda-\mu_1} + \frac{\lambda K}{\alpha+\lambda}, \quad s > 0,$$

and remark that $b > 0$ satisfies

$$-b = c_1 \beta^+ b^{\beta^+} + c_2(b)\beta^- b^{\beta^-} - \frac{\lambda b}{\alpha+\lambda-\mu_1}$$

if and only if

$$\frac{\mathrm{d}}{\mathrm{d}s} V_b(s)|_{s=b} = V_b'(b) = -1.$$

Therefore, when setting $\Gamma(b) = V_b'(b)$ for $b > 0$, the proof of the lemma reduces to showing that the equation $\Gamma(b) = -1$ has exactly one root between zero and K; and this will be shown next.

By straightforward calculation, we have that

$$\Gamma(b) = c_1 b^{\beta^+ - 1}(\beta^+ - \beta^-) + \frac{\alpha K \beta^-}{(\alpha+\lambda)b} - \frac{(\alpha-\mu_1)\beta^- + \lambda}{\alpha+\lambda-\mu_1} \quad \text{for} \quad b > 0$$

so that

$$\lim_{b\to 0}\Gamma(b) = -\infty \text{ (since } \beta^- < 0) \quad \text{and} \quad \Gamma(K) = 0.$$

Thus, by the Intermediate Value Theorem, there exists $b_1 \in (0, K)$ such that $\Gamma(b_1) = -1$.

For uniqueness, one only has to show that $\Gamma(\cdot)$ is increasing on $(0, K)$, that is, $\Gamma'(b) \neq 0$ for all $b \in (0, K)$.

But,

$$\Gamma'(b) = c_1(\beta^+ - 1)(\beta^+ - \beta^-) b^{\beta^+ - 2} - \frac{\alpha K \beta^-}{\alpha+\lambda} b^{-2},$$

and hence, for $b \in (0, K)$, the equality $\Gamma'(b) = 0$ is equivalent to

$$\left(\frac{b}{K}\right)^{\beta^+} = \frac{\alpha(\alpha+\lambda-\mu_1)}{\frac{1}{2}\sigma_1^2\lambda(\beta^+ - 1)(\beta^- - 1)},$$

where we have used that β^- is a root of the equation (3.6). As the above right-hand side is always negative because $\beta^+ > 1$ if $\alpha + \lambda > \mu_1$ and $\beta^+ < 1$ if $\alpha + \lambda < \mu_1$, there is no $b > 0$ such that $\Gamma'(b) = 0$. $\square$

4 On Optimal Stopping under Barndorff-Nielson Shephard Model

4.1 Introduction

4.1.1 Literature review and chapter summary

In their seminal paper [7], Barndorff-Nielsen and Shephard introduced a model that has been shown to describe the behaviour of certain financial assets processes particular well. Under this model, the squared volatility of a risky asset is described by an Ornstein-Uhlenbeck type process with a Lévy subordinator as the background driving process. A key feature of this model is that an upward jump in volatility of the underlying asset occurs at the same time as a downward jump in price. We will refer to this model as the BNS model. Moreover, Barndorff-Nielsen suggested in [6] that generalised inverse Gaussian distributions are suitable distributions for the Lévy process.

The BNS model has been studied from the point of view of portfolio selection in [8]. [54] is a study on the equivalent martingale measures and the European option pricing for the BNS model. The authors of [54] proposed a transform-based method and a simple Monte Carlo method. The only paper we have found on the subject of American options is [61]. [61] claims that the price of an American option under the BNS model is the unique viscosity solution of the standard variational inequality associated with optimal stopping problems, which was discussed at the beginning of Section 1.2.

We consider the problem of American option valuation under the BNS model. In particular, we are interested in the American put problem in both finite and infinite horizon. Our analysis uses a combination of probabilistic and analytical methods. We are able to prove monotonicity and continuity properties of option prices under the BNS model. From this, we can deduce that the stopping region of the optimal stopping problem is characterised by a monotone stopping boundary in infinite horizon and by a monotone stopping surface in finite horizon. In infinite horizon, the stopping boundary is continuous subject to some additional assumptions about the jump measure. In finite horizon, the continuity can only be proved in a specific direction. By appealing to classical results in PDE theory, we are able to strengthen the results in [61]. In infinite horizon, we show that the value function satisfies its corresponding variational inequality everywhere except on the stopping boundary. In finite horizon, this, too, holds subject to a change of variable.

From a theoretical point of view, the BNS model poses more difficult challenges than jump diffusion models and stochastic volatility models without jumps. Firstly, the underlying asset has jumps, hence the time change argument in [2] cannot be applied to prove monotonicity in volatility. Secondly, the squared volatility process has no diffusion component, so the PIDE operator in the variational inequality is no longer 'nice'. This means similar arguments to those appearing in [59] and [77] cannot be applied without modification, especially for finite horizon problems.

The practical implication of our regularity results is that discretisation techniques such as finite difference can be used to solve the variational inequality numerically to approximate the value function. Often, these regularity properties are assumed rather than proven when developing numerical methods. This was the case in [3], where the authors studied the Heston model with jumps.

The rest of this chapter is organised as follows. Section 4.1 introduces the probabilistic set-up of the BNS model and the relevant optimal stopping problems. In Section 4.2 and 4.3, we prove monotonicity properties of the value function and stopping boundary/surface. In Section 4.4 and Section 4.5, we prove some regularity results about the value function and continuity of the stopping boundary/surface for both infinite and horizon problem. Section 4.6 is a stand-alone section using a Monte Carlo method to estimate the value function to check some of the properties we proved in the previous sections. Section 4.7 contains the auxiliary proofs omitted in Section 4.1 - 4.6.

4.1.2 Probabilistic set-up for the BNS model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which supports two independent Lévy processes, a Brownian motion B_t and a Lévy subordinator Z_t, which is assumed to be driftless. Since Z is a Lévy subordinator, it has a Lévy measure Π with support on $(0, \infty)$, which satisfies the condition

$$\int_0^\infty (1 \wedge z)\Pi(\mathrm{d}z) < \infty. \tag{4.1}$$

Let $(\mathcal{F}_t)_{t \geq 0}$ be the complete augmented filtration generated by $(B_s, Z_{\lambda s} : s \leq t)$. We consider a pair of processes (X_t, V_t) driven by the following pair of SDEs

$$dX_t = \sqrt{V_t} dW_t - (\frac{1}{2}V_t - r - \lambda\Phi(\rho))dt - \rho dZ_{\lambda t}$$

$$dV_t = -\lambda V_t dt + dZ_{\lambda t} \tag{4.2}$$

where Φ is the Lévy symbol for Z. r, λ and ρ are positive constants. Usually, X_t model the log price of an underlying asset and V_t models the squared volatility. We denote the price of the asset by $S_t = \exp(X_t)$. Here Z is often referred to as the background driving Lévy process (BDLP).

We construct a family of these processes for initial values $S_0 > 0$ and $V_0 \geq 0$ on the probability space $\mathbb{P}$, driven by B and Z. On such a space, the pair (S, V) is strong Markov with state space $\mathcal{O} \stackrel{\text{def}}{=} (0, \infty) \times [0, \infty)$.

We use $S^{s,v}$ to denote the process S with initial conditions $S_0 = s$ and $V_0 = v$. Similarly, $X^{x,v}$ denotes the process X with the initial condition $X_0 = x$ and $V_0 = v$ and V^v denotes the process V with the initial condition $V_0 = v$. We can use the notation V^v because the evolution of the V does not depend on S or X and it can be shown that V is a strong Markov process.

Remark 4.1. We note that under this set up, $e^{-rt}S_t$ is a martingale. This is particularly relevant in financial applications, where $\mathbb{P}$ is known as a martingale measure (or a risk-neutral measure). This is a condition we impose to ensure that there is no arbitrage in the model. Notice that if the drift term of X is changed so that $e^{-rt}S_t$ is no longer a martingale, a change of measure is required such that $e^{-rt}S_t$ is a martingale under the new measure. In some cases, the dynamics of (S, V) does not even follow a different BNS model under the changed measure. We avoid this by effectively choosing a risk neutral measure to work with. For a discussion of equivalent martingale measures for the BNS model, see [54].

We can now give a financial interpretation for the parameters appearing in the BNS model. In this risk-neutral set up, r can be interpreted as the risk-free rate. It is implicitly assumed that investors can invest in a riskless asset as well as the asset with price process S. When investing in this riskless asset, an investor can expect 1 unit of money at 0 to return e^{rt} unit of money at time t with certainty. λ is the rate the square-volatility V decays at. It is assume that V decays exponentially when there are no jumps in the price of the underlying asset. ρ can be regarded as a measure of correlation between the log-price X and V. When the squared volatility increases by 1 unit, the log-price X decreases by ρ units. This feature is intended to capture some aspects of the leverage effect, which we discussed when introducing our interactive volatility model in Chapter 3. We refer to Section 3.1 for a more detailed discussion.

Under the BNS model, the process V is an Ornstein-Uhlenbeck process driven by a subordinator and therefore admits the representation

$$V_t^v = V_{t'}^v e^{-\lambda(t-t')} + e^{-\lambda(t-t')} \int_{t'}^{t} e^{\lambda u} dZ_{\lambda u} \qquad \text{for } t > t'. \tag{4.3}$$

Alternatively, it can be written as

$$V_t^v = V_{t'}^v e^{-\lambda(t-t')} + \sum_{t'<u\leq t} \Delta Z_u e^{-\lambda(t-u)} \qquad \text{for } t > t', \tag{4.4}$$

where $\Delta Z_u = Z_u - Z_{u-}$ is the jump at time u.

Moreover, $X_t^{x,v}$ has the explicit representation

$$X_t^{x,v} = X_{t'}^{x,v} + (r+\lambda\Phi(\rho))(t-t') - \rho(Z_{\lambda t} - Z_{\lambda t'}) + \int_s^t \sqrt{V_u^v} \mathrm{d}W_u - \frac{1}{2}\int_s^t V_u^v \mathrm{d}u \quad \text{for } t > t', \tag{4.5}$$

so conditional on $X_{t'}^{x,v}$, $Z_{\lambda t} - Z_{\lambda t'}$ and $(V_u : t' \leq u \leq t)$, $X_t^{x,v}$ is normally distributed.

Exponentiating (4.5), we see that $S_t^{s,v}$ is lognormally distributed when conditioned on $S_{t'}^{s,v}$, $Z_{\lambda t} - Z_{\lambda t'}$ and $(V_u : t' \leq u \leq t)$.

Remark 4.2. (i) By setting $t' = 0$ in (4.3), we have

$$V_t^v = ve^{-\lambda t} + e^{-\lambda t} \int_0^t e^{\lambda q} dZ_{\lambda q},$$

This means for $\epsilon > 0$, $t \geq 0$, $V_t^{v+\epsilon} \geq V_t^v$ and the following equation holds:

$$V_t^{v+\epsilon} = V_t^v + \epsilon e^{-\lambda t}. \tag{4.6}$$

(ii) Under the BNS model, 0 is a transient state. If $V_0 = 0$, $P(V_t > 0) > 0$ for all $t > 0$. However, if $V_0 > 0$, then $P(V_t = 0) = 0$. In other models, the state 0 is often absorbing state; in which case, the value of option at $v = 0$ is often known. The transience property causes extra difficulty in numerical schemes, see Remark 4.29.

An important tool available to us is the variational inequality. In order to know the correct variational inequality satisfied by the value function, we need the infinitesimal generator L of the pair (S, V). The infinitesimal generator of (S, V) acting on C^2 functions is given by

$$\begin{aligned} Lf(s,v) = \frac{1}{2}s^2 v \partial_{11} f(s,v) &+ rs\partial_1 f(s,v) - \lambda v \partial_2 f(s,v) \\ &+ \lambda \int_0^\infty f(se^{-\rho z}, v+z) - f(s,v) + s(1-e^{-\rho z})\partial_1 f(s,v)\Pi(\mathrm{d}z). \end{aligned}$$

The derivation of this generator is found on page 131 in the Chapter Appendix. By a similar calculation, it is possible to show the process V is also a strong Markov process with the infinitesimal generator (acting on C^1 functions)

$$L_V f(v) = -\lambda v f'(v) + \lambda \int_0^\infty f(v+z) - f(v)\Pi(\mathrm{d}z) \quad \text{for } v \in [0,\infty).$$

4.1.3 Definition of some optimal stopping problems under the BNS model

For a given pay-off function g, the price of the American option with this pay-off is found by solving the following optimal stopping problem:

$$u(s,v,T) = \sup_{0 \leq \tau \leq T} \mathbb{E} e^{-r\tau} g(S_\tau^{s,v}) \quad \text{for } s > 0, v \geq 0, T \in [0,\infty],$$

where the supremum is taken over all stopping times. As introduced in Chapter 1, u is referred to as the value function and g is referred to as the pay-off or gain function.

Throughout the rest of this chapter, **we assume that g is a convex function**. An important pay-off function which satisfies this condition is $(K-\cdot)^+$, which is the pay-off of a put option.

Definition 4.3. *Let $g(\cdot):(0,\infty)\to\mathbb{R}$ be a convex function. We define $u:(0,\infty)\times[0,\infty)\times[0,\infty]\to(0,\infty)$ as*

$$u(s,v,T)=\sup_{0\le\tau\le T}\mathbb{E}e^{-r\tau}g(S_\tau^{s,v})\quad \text{for } s>0, v\ge 0, T\in[0,\infty], \tag{4.7}$$

where the supremum is taken over the set of all stopping time with respect to the filtration $\mathcal{F}_t$ bounded above by T. When $T=\infty$, the value function is independent of T and we use the shorthand $u(s,v)=u(s,v,\infty)$.

To ensure the value function is well defined, we impose the following condition.

Assumption 4.4. *We assume that $\mathbb{E}\sup\limits_{0\le t\le T}e^{-rt}|g(S_t^{s,v})|<\infty$ for all $s>0, v\ge 0$.*

It is trivial to see that this condition is automatically satisfied when the gain function is bounded. A closely related object to American options is Bermudan options, where the option holder can only exercise the option at some pre-agreed fixed times. We now define the value function for a Bermudan option with equally spaced exercise times.

Definition 4.5. *Let $g(\cdot):(0,\infty)\to\mathbb{R}$ be a convex function. We define $U:(0,\infty)\times[0,\infty)\times[0,\infty)\times\mathbb{N}^+\to(0,\infty)$ as*

$$U(s,v,T,n)=\sup_{\tau\in\{\frac{T}{n},\frac{2T}{n},\dots,T\}}\mathbb{E}e^{-r\tau}g(S_\tau^{s,v})\quad \text{for } s>0, v\ge 0, T\in[0,\infty), n\in\mathbb{N}^+. \tag{4.8}$$

For a given n, the set of permitted exercise times for the Bermudan option in the definition above is $\{\frac{T}{n},\frac{2T}{n},...,T\}$. In particular, $U(s,v,T,1)=\mathbb{E}e^{-rT}g(S_T^{s,v})$ is the price of the European option with expiry T.

4.2 General properties of value functions for convex gain function

4.2.1 Limit relationship between value functions of Bermudan and American options

Proposition 4.6. *Let $u(s,v,T)$ denote the price of an American option under the BNS model with $S_0=s$, $V_0=v$ and expiry T, then*

$$\lim_{m\to\infty}U(s,v,T,2^m)=u(s,v,T) \tag{4.9}$$

The proof of this proposition is found in the Chapter Appendix on page 129. This lemma is relatively standard and its proof similar to that of Lemma 2.21.

Remark 4.7. Pricing Bermudan options is an interesting problem in its own right, but there are two reasons why they are relevant from the point of view of American options.

The first reason is theoretical. Proposition 4.6 tells us that the price of Bermudan options converges monotonically to the price of the corresponding American options with the same pay-off if nested sequence of permitted exercise times are chosen. It is sometimes easier to study Bermudan options because fixed exercise times allow us to use the transition kernel of the processes (S, V). The results in the rest of this subsection are proved first for Bermudan options and then proved for American options by taking limits.

In comparison, tackling the American option problem directly requires understanding of the expected value of the gain process evaluated at stopping times. The current success seems to be restricted to relatively simple one dimensional processes in infinite time horizon. For example, see [1] for pricing American options when the price process is a spectrally one-sided Lévy processes.

The second reasons is a practical one. Some numerical schemes for solving American option problems usually price the Bermudan option instead and give a bound on the price gap. This means instead of pricing American options, we are actually pricing Bermudan options to approximate the American option prices. This is the method used in Section 4.6 to obtain numerical results for American put options under the BNS model.

4.2.2 Convexity and monotonicity results

Having now established the connection between Bermudan and American options, we proceed according to the plan set out in Remark 4.7. The goal of this section is to prove the convexity of the value function with respect to the price variable and the monotonicity with respect to variance variable for the value functions of Bermudan and American options.

The results in this section not only help us to gain better understanding about the value function, but also provide us with information about what the stopping region looks like. They are important for the results we prove about the shape of the stopping region of the American put problem in the next section.

Proposition 4.8. *Recall the definition of u and U given by (4.7) and (4.8), then the following statements hold for $T \in [0, \infty)$.*

(i) $u(s, v, T)$ and $U(s, v, T, n)$ are convex in the price variable s.

(ii) $u(s, v, T)$ and $U(s, v, T, n)$ are increasing in variance variable v.

In addition, if the condition

$$u(s,v,\infty)=\lim_{T\to\infty}u(s,v,T) \tag{4.10}$$

holds, then (i) and (ii) also holds for $u(s,v)$.

The assumption given by (4.10) does not always hold, but it holds if the gain function is bounded (as it is the case in the perpetual American put problem).

Lemma 4.9. *Consider the set-up given in Definition 4.3 with the additional assumption that g is bounded, then u satisfies condition (4.10).*

See page 130 for a proof of this lemma. We now prove Proposition 4.8. Our strategy is to first prove Proposition 4.8 for $U(s,v,T,n)$ by induction, then Proposition 4.8 also holds for $u(s,v,T)$ by taking appropriate limits. The following lemma is the base case of the induction.

Lemma 4.10. *The price of the European option $U(s,v,T,1)$ defined in (4.8) is convex in s and increasing in v for $s>0, v\geq 0$.*

Proof. We introduce the following notations: if Y has a log-normal distribution with $\log(Y)\sim N(-\frac{1}{2}y,y)$, we denote the distribution function of Y by N_y. The use Y_1 and Y_2 to denote the integrated square volatility for the two cases $V_0=v$ and $V_0=v+\epsilon$, where

$$Y_2=\int_0^T V_q^v\,\mathrm{d}q \quad\text{and}\quad Y_1=\int_0^T V_q^{v+\epsilon}\,\mathrm{d}q.$$

By Remark 4.2, $Y_1\geq Y_2$ almost surely. Furthermore, we denote the σ-algebra generated by Z up to the expiry time T by $\sigma(Z_{\lambda q}:q\leq T)$.

First, we prove $U(s,\cdot,T,1)$ is increasing. We observe that $S_T^{s,v}$ can be written as,

$$S_T^{s,v}=s\exp\Big((r+\lambda\Phi(\rho))T-\rho Z_{\lambda T}\Big)\underbrace{\exp\Big(\int_0^T\sqrt{V_t}\mathrm{d}W_t-\frac{1}{2}\int_0^T V_t\mathrm{d}t\Big)}. \tag{4.11}$$

Note the braced part of equation (4.11) has expectation 1 and its conditional distribution with respect to $\sigma(Z_{\lambda q}:q\leq T)$ is log-normal. We now proceed by the law of total expectation. For any $\epsilon>0$, we obtain

$$\begin{aligned}
U(s,v+\epsilon,&T,1)-U(s,v,T,1)\\
&=\mathbb{E}[e^{-rT}(g(S_T^{s,v+\epsilon})-g(S_T^{s,v}))]\\
&=\mathbb{E}[\mathbb{E}[e^{-rT}(g(S_T^{s,v+\epsilon})-g(S_T^{s,v}))|\sigma(Z_{\lambda q}:q\leq T)]]\\
&=\mathbb{E}[\underbrace{\mathbb{E}[e^{-rT}(g(sc\eta_1)-g(sc\eta_2))]}_{(4.13)}|_{\nu_q=V_q^{v+\epsilon},\mu_q=V_q^v,c=e^{(r+\lambda\Phi(\rho))T-\rho Z_{\lambda T}}}],
\end{aligned} \tag{4.12}$$

where η_i has log-normal distribution with distribution functions N_{y_i} for $i = 1, 2$. y_i are values of Y_i when conditioned with respect to $\sigma(Z_{\lambda q} : q \leq T)$. Explicitly, they are given by

$$y_2 = \int_0^T \mu_q \, \mathrm{d}q \quad \text{and} \quad y_1 = \int_0^T \nu_q \, \mathrm{d}q = \int_0^T \mu_q + \epsilon e^{-\lambda q} \, \mathrm{d}q = y_2 + \frac{\epsilon}{\lambda}(1 - e^{-\lambda T}),$$

where the expression for y_1 in term of y_2 follows from (4.6). This allows us to write the braced part of (4.12) as the following integral:

$$\int_0^\infty g(scp) N_{y_1}(\mathrm{d}p) - \int_0^\infty g(scp) N_{y_2}(\mathrm{d}p). \tag{4.13}$$

Now consider an independent probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ supporting two independent random variables ζ_2 and η with distribution function N_{y_2} and $N_{(y_1 - y_2)}$. Let $\log(\zeta_1) = \log(\zeta_2) + \log(\eta)$. Note that $\log(\zeta_1) \sim N(-\frac{1}{2}y_1, y_1)$, so ζ_1 has density function N_{y_1}. We have, for any $s > 0, c > 0, y_1 > y_2$,

$$\begin{aligned}
\int_0^\infty g(scx) N_{y_1}(\mathrm{d}x) - \int_0^\infty g(scx) N_{y_2}(\mathrm{d}x) &= \hat{\mathbb{E}}[e^{-rT}(g(sc\zeta_1) - g(sc\zeta_2)] \\
&= \hat{\mathbb{E}}[e^{-rT}(g(sc\zeta_2\eta) - g(sc\zeta_2)] \\
&= \hat{\mathbb{E}}[\hat{\mathbb{E}}[e^{-rT}(g(sc\zeta_2\eta) - g(sc\zeta_2) | \sigma(\zeta_2)]] \\
&= \hat{\mathbb{E}}[\underbrace{\hat{\mathbb{E}}[e^{-rT}(g(scu\eta) - g(scu)]}_{\geq 0 \text{ by Jensen's inequality}} |_{u=\zeta_2}] \geq 0
\end{aligned}$$

Note that Jensen's inequality applies since $\hat{\mathbb{E}}\eta = 1$ and g is convex. It follows that the integral denoted by (4.13) is non-negative and $U(s, v + \epsilon, T, 1) \geq U(s, v, T, 1)$ from (4.12).

We now show convexity of $U(\cdot, v, T, 1)$. It is clear from (4.11) that $S^{s,v} = sS^{1,v}$, so for $t \in (0, 1)$, we have:

$$\begin{aligned}
U(ts_1 + (1-t)s_2, v, T, 1) &= \mathbb{E}e^{-rT} g(S_T^{ts_1 + (1-t)s_2, v}) \\
&= \mathbb{E}e^{-rT} g((ts_1 + (1-t)s_2) S_T^{1,v}) \\
&= \mathbb{E}e^{-rT} g(ts_1 S_T^{1,v} + (1-t)s_2 S_T^{1,v}) \\
&\leq \mathbb{E}e^{-rT}(tg(s_1 S_T^{1,v}) + (1-t)g(s_2 S_T^{1,v})) \\
&= t\mathbb{E}e^{-rT} g(s_1 S_T^{1,v}) + (1-t)\mathbb{E}e^{-rT} g(s_2 S_T^{1,v}) \\
&= t\mathbb{E}e^{-rT} g(S_T^{s_1,v}) + (1-t)\mathbb{E}e^{-rT} g(S_T^{s_2,v}) \\
&= tU(s_1, v, T, 1) + (1-t)U(s_2, v, T, 1),
\end{aligned}$$

where the inequality holds because g is a convex function. □

Before we proceed to our next result, we make the following observation. By Bellman-Wald Equation, we have the following representation:

$$U(s,v,T,n) = e^{-rT/n}\mathbb{E}[\max(g(S^{s,v}_{\frac{T}{n}}), U(S^{s,v}_{\frac{T}{n}}, V^{v}_{\frac{T}{n}}, \tfrac{T(n-1)}{n}, n-1))] \tag{4.14}$$

We now prove that $U(s,v,T,n)$ is convex in s and increasing in v for every n and finite T by induction.

Proposition 4.11 (Proposition 4.8 for U)**.** *Under the BNS model, $U(s,v,T,n)$ defined in (4.8) is increasing in v and convex in s for all $s>0, v\geq 0$ and $n\in\mathbb{N}$.*

Proof. We proceed by induction. We proved the $n=1$ case in Lemma 4.10. For $n\geq 2$, assume the proposition holds for $n-1$. For this particular value of T and n, define the function

$$w(s,v) = \max(g(s),\ U(s,v,\tfrac{T(n-1)}{n}, n-1))$$

Note $U(s,v,\frac{T(n-1)}{n},n-1)$ is increasing in v and convex in s by induction hypothesis, so w is also increasing in v and convex in s. Hence, by (4.14), we have that

$$\begin{aligned}
&U(s,v+\epsilon,T,n) - U(s,v,T,n)\\
=&\mathbb{E}e^{-rT/n}\Bigg[\underbrace{w(S^{s,v+\epsilon}_{T/n}, V^{v+\epsilon}_{T/n}) - w(S^{s,v+\epsilon}_{T/n}, V^{v}_{T/n})}_{\geq 0 \text{ since } w(s,\cdot) \text{ is increasing}}\Bigg] + \mathbb{E}e^{-rT/n}\Bigg[w(S^{s,v+\epsilon}_{T/n}, V^{v}_{T/n}) - w(S^{s,v}_{T/n}, V^{v}_{T/n})\Bigg]\\
\geq&\mathbb{E}e^{-rT/n}\Bigg[w(S^{s,v+\epsilon}_{T/n}, V^{v}_{T/n}) - w(S^{s,v}_{T/n}, V^{v}_{T/n})\Bigg]
\end{aligned}$$

The remaining part of the argument similar to the case $n=1$,

$$\begin{aligned}
&\mathbb{E}e^{-rT/n}\Bigg[w(S^{s,v+\epsilon}_{T/n}, V^{v}_{T/n}) - w(S^{s,v}_{T/n}, V^{v}_{T/n})\Bigg]\\
&\quad= \mathbb{E}[\mathbb{E}[e^{-rT}(w(S^{s,v+\epsilon}_{T}, V^{v}_{\frac{T}{n}}) - w(S^{s,v}_{T}, V^{v}_{\frac{T}{n}}))|\sigma(Z_{\lambda q}: q\leq \tfrac{T}{n})]]\\
&\quad= \mathbb{E}\Bigg[\underbrace{\mathbb{E}\Bigg[e^{-rT}(w(sc\eta_1,\mu_q) - w(sc\eta_2,\mu_q))\Bigg]}_{(4.16)}\Bigg|_{\nu_q=V^{v+\epsilon}_q, \mu_q=V^{v}_q, c=e^{(r+\lambda\Phi(\rho))\frac{T}{n}-\rho Z_{\lambda T/n}}}\Bigg],
\end{aligned} \tag{4.15}$$

where η_i has log-normal distribution with distribution functions N_{y_i} for $i=1,2$. y_i are integrated square volatility up on conditioning with respect to $\sigma(Z_{\lambda q}: q\leq T/n)$. Explicitly, they are given by

$$y_2 = \int_0^{T/n} \mu_q \,\mathrm{d}q \;\text{ and }\; y_1 = \int_0^{T/n} \nu_q \,\mathrm{d}q = \int_0^{T/n} \mu_q + \epsilon e^{-\lambda q}\,\mathrm{d}q = y_2 + \frac{\epsilon}{\lambda}(1-e^{-\lambda T/n}),$$

where the expression for y_1 in term of y_2 follows from (4.6). This allows us to write the braced expression in (4.15) as the following integral:

$$\int_0^\infty w(scp, \mu_q) N_{y_1}(\mathrm{d}p) - \int_0^\infty w(scp, \mu_q) N_{y_2}(\mathrm{d}p). \tag{4.16}$$

By an identical argument to the one following (4.13), except swapping $g(\cdot)$ for $w(\cdot, \mu_q)$, we have that

$$\int_0^\infty w(scp, \mu_q) N_{y_1}(\mathrm{d}p) - \int_0^\infty w(scp, \mu_q) N_{y_2}(\mathrm{d}p) \geq 0,$$

hence $U(s, v, T, n)$ is increasing in v, if $U(s, v, T, n-1)$ is increasing in v and convex in s. The argument for convexity in s is similar to the argument used to show $U(\cdot, v, T, 1)$ is convex in Lemma 4.10. For $t \in (0, 1)$, we have that

$$\begin{aligned} U(ts_1 + (1-t)s_2, v, T, n) =& \mathbb{E} e^{-rT/n}[w(S_{T/n}^{ts_1+(1-t)s_2,v}, V_{T/n})] \\ =& \mathbb{E} e^{-rT/n}[w((ts_1 + (1-t)s_2) S_{T/n}^{1,v}, V_{T/n})] \\ \leq& \mathbb{E} e^{-rT/n}[tw(s_1 S_{T/n}^{1,v},, V_{T/n}) + (1-t) w(s_2 S_{T/n}^{1,v}, V_{T/n}^{v})] \\ =& tU(s_1, v, T, n) + (1-t) U(s_2, v, T, n), \end{aligned}$$

where the inequality holds by convexity of $w(\cdot, v)$. □

Proposition 4.8 holds for u as a corollary.

Corollary 4.12 (Proposition 4.8 for u)**.** *$u(s, v, T)$ is convex in s and increasing in v for all $T \in [0, \infty)$. This also holds for $T = \infty$ if (4.10) holds.*

Proof. The short proof for this result is structured as follows.

(i) We show that $u(s, v, T)$ is convex in s for a finite T.

Proposition 4.11 tells us that $U(\cdot, v, T, 2^m)$ is convex. Moreover, by Proposition 4.6, for a finite T, $u(s, v, T)$ is the pointwise limit of $U(s, v, T, 2^m)$ as m tends to the infinity. Since the limit of a convex function is convex, we conclude that $u(\cdot, v, T)$ is convex.

(ii) We show that $u(s, v, T)$ is convex in s for $T = \infty$ provided (4.10) is satisfied.

For $T = \infty$, by (4.10), we have that

$$\lim_{T \to \infty} u(s, v, T) = u(s, v, \infty) = u(s, v),$$

Since $u(s, v, T)$ is shown to be convex in (i), we see that $u(s, v, \infty)$ is convex as it is the limit of convex functions.

(iii) We show that $u(s,v,T)$ is increasing in v for all $T \in (0,\infty)$ and $T=\infty$ if (4.10) holds. The argument in step (i) and (ii) can be repeated for monotonicity in v because like convexity, monotonicity is preserved by taking limit in m and T.

□

Remark 4.13. (i) The proof we used for Proposition 4.8 still works if we change the linear drift term of the process X_t from $(r+\Phi(\rho))\mathrm{d}t$ to $c\mathrm{d}t$ for any constant c, but we cannot change the non-constant term $-\frac{1}{2}V_t\mathrm{d}t$. We use the fact that conditional distribution of $\sqrt{V_t}\mathrm{d}B_t - \frac{1}{2}V_t\mathrm{d}t$ is log-normal and found a coupling to take advantage of this.

(ii) The proof of Proposition 4.8 is similar to the proof of Proposition 2.22. In general, the comparison method works for the class of model driven by Markov processes $B_t, V_t, \tilde{V}_t, M_t$ such that

$$dX_t = \sqrt{V_t}dB_t - \frac{1}{2}V_t\mathrm{d}t + \mathrm{d}M_t,$$

$$dX_t = \sqrt{\tilde{V}_t}dB_t - \frac{1}{2}\tilde{V}_t\mathrm{d}t + \mathrm{d}M_t,$$

where B is a Brownian motion, M is a Markov process and the distribution of M_t conditional on $\sigma(V_s : s \leq t)$ is the same as the distribution of M_t conditional on $\sigma(\tilde{V}_s : s \leq t)$.

$$V_t \geq \tilde{V}_t \quad \text{for all } t \geq 0.$$

4.3 Characterisation of the stopping region for the American put problem

The result in the rest of this chapter are predominantly related to American put problem where $g(s) = (K-s)^+$. The key results of this section are Proposition 4.20 and Proposition 4.22, where we characterise the stopping region of the American problem by a stopping boundary and shows its left-continuity.

We begin by giving two definition related to the stopping region.

Definition 4.14. *The stopping region, D, of the optimal stopping problem is defined by*

$$D \stackrel{def}{=} \{(s,v,T) : u(s,v,T) = g(s)\} \tag{4.17}$$

Another object relevant to the American put is the maximum price at which the option should be exercised for a given value of v and T.

Definition 4.15. *The function $b : [0,\infty) \times [0,\infty] \to [0,\infty)$ is defined by*

$$b(v,T) \stackrel{def}{=} \sup\{s : u(s,v,T) = (K-s)^+\}. \tag{4.18}$$

In addition, we use $b(v)$ as a shorthand for $b(v,\infty)$. We use the convention that $\sup\emptyset = 0$.

Remark 4.16. It makes sense to use the convention that when the right side of equation (4.18) is an empty set, we take $b(v,T) = 0$. We can in fact extend the definition of $u(s,v,T)$ fo $s = 0$. In this case,

$$K = g(0) \le u(0,v,T) \le \sup_{s\ge 0} g(s) = K, \tag{4.19}$$

so 0 is always in the set on right hand side of (4.18) in this extended definition of u. The convention $\sup\emptyset = 0$ is consistent with this extended definition of value function. However, it is possible to show that $b(v,T) \neq 0$ any value of v and T.

In this section, we first prove two monotonicity lemmas related to the value function with respect to the price parameter s. Combining these new results with the monotonicity result in the variance parameter v in the previous section, we show that the stopping region D can be characterised entirely by b.

Lemma 4.17. *$u(\cdot,v,T)$ and $U(\cdot,v,T,n)$ is a decreasing function for $v > 0, T \in [0,\infty]$.*

Proof. Using the inequality $a^+ - b^+ \le (a-b)^+$, we have

$$\begin{aligned}
u(s+\epsilon,v,T) - u(s,v,T) &= \sup_{0\le\tau\le T} \mathbb{E}[e^{-r\tau}(K - S_\tau^{s+\epsilon,v})^+] - \sup_{0\le\tau\le T} \mathbb{E}[e^{-r\tau}(K - S_\tau^{s,v})^+] \\
&\le \sup_{0\le\tau\le T} \mathbb{E}[e^{-r\tau}((K - S_\tau^{s+\epsilon,v})^+ - (K - S_\tau^{s,v})^+)] \\
&\le \sup_{0\le\tau\le T} \mathbb{E}[e^{-r\tau}(S_\tau^{s,v} - S_\tau^{s+\epsilon,v})^+] = 0,
\end{aligned}$$

where the last equality holds since $S_\tau^{s,v} \le S_\tau^{s+\epsilon,v}$ almost surely. The proof for U works exactly the same way. □

Lemma 4.18. *For fixed $v \ge 0$, $T \in [0,\infty]$, the function*

$$s \mapsto u(s,v,T) - g(s)$$

is non-decreasing in $[0,K]$.

Proof. We first prove this lemma for $T < \infty$. For any s,ϵ such that $0 < s < s+\epsilon < K$, let $\tau \le T$ to be a stopping time such that

$$u(s,v,T) = \mathbb{E}e^{-r\tau}(K - S_\tau^{s+\epsilon,v})^+.$$

Note, $\mathbb{E}[e^{-r\tau}S_\tau^{s,v}] = s$ since τ is a bounded stopping time. Then, we have that

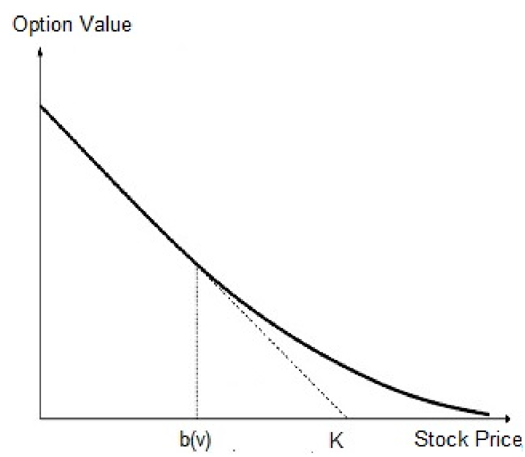
$$\begin{aligned}
u(s+\epsilon, v, T) &= \mathbb{E}[e^{-r\tau}(K - S_\tau^{s+\epsilon,v})^+] \\
&\geq \mathbb{E}[e^{-r\tau}(K - S_\tau^{s,v})^+] + \mathbb{E}[(e^{-r\tau}S_\tau^{s+\epsilon,v} - e^{-r\tau}S_\tau^{s,v})^+] \\
&\geq u(s,v,T) - \left(\frac{s+\epsilon}{s} - 1\right)\mathbb{E}[e^{-r\tau}S_\tau^{s,v}] \\
&= u(s,v,T) - (s+\epsilon) + s \\
&= u(s,v,T) - g(s) + g(s+\epsilon),
\end{aligned}$$

where the second line follows from the first by the inequality $a^+ - b^+ \leq (a-b)^+$. For $T = \infty$, we just need to take the limit $T \to \infty$ on the result for $T < \infty$. □

$u(s,v,T) - g(s)$ is the premium the current option value has over its current exercise value. When this is equal to 0, the option should be exercised. By Remark 4.16, we know that $g(b(v,T)) - u(b(v,T),v,T) = 0$. Lemma 4.18 tells us that

$$u(s,v,T) = g(s) \quad \text{for} \quad s \leq b(v,T).$$

Combining this with the fact u is convex and decreasing in s, we know the value $u(\cdot, v, T)$ looks like



An illustration of value function for a fixed value of v

We now show that the stopping region D can be completely characterised by the stopping boundary b. In order to prove this, we need the following lemma.

Lemma 4.19. *Recall the definition of $b(v,T)$ given by (4.17). The following holds*

(i) $u(b(v,T),v,T) = (K - b(v,T))^+$

(ii) For $T > 0, v > 0$, $b(v,T) < K$.

Proof. By definition of b, there exists a sequence s_n such that $u(s_n, v, T) = (K - s_n)^+$ and $s_n \uparrow b(v, T)$. Since u is convex in s and convex functions are continuous, $u(\cdot, v, T)$ is continuous, so we have

$$u(b(v,T), v, T) = \lim_{n\to\infty} u(s_n, v, T) = \lim_{n\to\infty} (K - s_n)^+ = (K - b(v,T))^+. \tag{4.20}$$

This prove (i). For (ii), suppose that $b(v, T) \geq K$, so $u(b(v, T), v, T) = 0$ by (i). Observe that for any $0 < t < T$, $\mathbb{P}(S_t^{s,v} < K/2) > 0$ for all $s, v > 0$. By choosing the stopping time to be t, we have

$$u(b(v,T), v, T) \geq \mathbb{E}e^{-rt}(K - S_t^{b(v),v})^+ \geq \tfrac{Ke^{-rt}}{2}\mathbb{P}(S_t^{s,v} < K/2) > 0,$$

which contradicts $u(b(v,T), v, T) = 0$. □

We are now ready to show that the stopping region D only consists of (s, v, T) such that $s \leq b(v, T)$.

Proposition 4.20. *Recall the definition of D given by* (4.18)*. We have that*

$$D = \{(s, v, T) : s \leq b(v, T)\}, \tag{4.21}$$

Proof. Firstly, by the definition of $b(v, T)$, we have

$$D \subseteq \{(s, v, T) : s \leq b(v, T)\}.$$

Secondly, because $u(s, v, 0) = g(s) = (K - s)^+$, so $b(v, 0) = \infty$ for all $v > 0$ and (4.21) holds for $T = 0$. Thirdly, for $T > 0$, Lemma 4.19 tells us that $b(v, T) < K$ and $u(b(v, T), v, T) = K - b(v, T)$. By Lemma 4.18, we see that

$$u(s, v, T) - (K - s) \leq u(b(v,T), v, T) - (K - b(v,T)) = 0 \quad \text{ for } s < b(v,T) < K,$$

so $u(s, v, T) \leq (K - s)$. However, $u(s, v, T) \geq g(s) = (K - s)$ by the definition of u, so

$$\{(s, v, T) : s \leq b(v, T)\} \subseteq D,$$

which allows us to conclude that (4.21) holds. □

Since the stopping region consists of values of s lying below $b(v, T)$, we refer to $b(v, T)$ (or $b(v)$ in the perpetual case) as the **stopping surface** (or **stopping boundary** in the perpetual case).

The next corollary follows immediately from Proposition 4.17 and Proposition 4.12. The proof is found on 133 in chapter appendix.

Corollary 4.21. *The left and right derivatives of u with respect to s exist and satisfy the conditions*

$$-1 \leq \partial_1^- u(s,v,T) \leq 0 \quad \text{for } s>0 \quad \text{and} \quad -1 \leq \partial_1^+ u(s,v,T) \leq 0 \quad \text{for } s \geq 0.$$

The same holds if $T = \infty$.

We now prove some properties of b without any additional assumptions. Namely, b is monotone and left continuous.

Proposition 4.22. *b has the following properties:*

(i) b is decreasing in both v and T,

(ii) b is jointly left continuous in (v,T), that is,

$$b(v^*,T^*) = \lim_{(v,T)\uparrow(v^*,T^*)} b(v,T)$$

Proof. For (i), consider $v' \geq v$ and $T' \geq T$, for all $s > b(v,T)$, we have that

$$u(s,v',T') \geq u(s,v,T) > g(s).$$

Since $u(b(v',T'),v',T') = g(b(v',T'))$ by (4.20), we can conclude that $b(v',T') \neq s$ for all $s > b(v,T)$. This means $b(v',T') \leq b(v,T)$, from which we conclude that b is decreasing in v and T.

For (ii), fix (v^*,T^*), by part (i), since b is decreasing in both variables, we have that

$$b^* \stackrel{\text{def}}{=} \lim_{(v,T),\uparrow(v^*,T^*)} b(v,T) \geq b(v^*,T^*) \tag{4.22}$$

For $v < v^*$ and $T < T^*$, by the continuity of u in the stopping region, we have that

$$u(b^*,v^*,T^*) = \lim_{(v,T)\uparrow(v^*,T^*)} u(b^*,v,T) = K - b^*, \tag{4.23}$$

where the second equality follows by the fact $b^* \leq b(v,T)$ for all $v < v^*, T < T^*$. (4.23) means that $b^* \leq b(v^*,T^*)$. Together with (4.22), this proves b is left continuous in (v,T). ∎

Remark 4.23. For $T = \infty$, we have that $b(v)$ is decreasing and left-continuous in v by the same argument.

The left continuity of the stopping boundary or stopping surface b holds without any additional assumptions. Instead of proving this fact directly, it is possible to show the left continuity of b must follow from the fact the stopping region is a closed set but it would require us to show that u is upper semi-continuous. See the discussion preceding Theorem 1.3 on page 8 or [57, pp.36] for further details. In the next section, we will show that the stopping boundary in the perpetual case is continuous under some additional assumptions.

4.4 Regularity property of value function and stopping boundary for the perpetual American put problem

Having now established the nature of the stopping region for the American put problem. The goal of this section is to gain better understanding of the regularity properties of the value function and stopping boundary. We will show that the value function and the stopping boundary are well behaved under suitable conditions. This may be able to provide justifications for using IPDE based numerical methods.

Our methods of showing the regularity property of the value function has three steps. The first step is to show the value function is locally Hölder continuous. The second step is to prove the viscosity solution property of the value function. The last step is using classical PDE theory to obtain higher order regularity than those first derived. The regularity of the value function is then used to prove continuity of the stopping boundary under appropriate assumptions.

4.4.1 Hölder continuity of the value function

We now prove a Hölder continuity property of $u(s, v, T)$. To do so, we first need to present two preliminary results. For $x = \log(s)$, we define $\tilde{u}(x, v, T) : (-\infty, \infty) \times (0, \infty) \times [0, \infty]$ by

$$\tilde{u}(x, v, T) \stackrel{\text{def}}{=} \tilde{u}(\log(s), v, T) = u(s, v, T). \tag{4.24}$$

We state a modified version of Proposition 3.3 in [61]. In [61], the constant C in the following proposition is dependent on T, but the proof can be modified in a way such that C is not dependent on T. Hölder continuity properties of u and $\tilde{u}$ follow from Proposition 4.24 and Proposition 4.25.

Proposition 4.24. *For $T \in [0, \infty]$, $\tilde{u}$ satisfies the condition*

$$|\tilde{u}(x, v, T) - \tilde{u}(x', v', T)| \leq C(|x - x'| + |v - v'| + |v - v'|^{1/2}),$$

where C is a constant independent of T.

The proof of Proposition 4.24 is found on page 134.

Proposition 4.25. *For $T \in [0, \infty]$, u satisfies*

$$|u(s, v, T) - u(s', v', T)| \leq C(|s - s'| + |v - v'| + |v - v'|^{\frac{1}{2}}),$$

where C is a constant independent of s, v and T.

Proof. We first show that, for fixed $v > 0$, $T \in [0, \infty]$ and $s, s' \in (0, \infty)$,

$$|u(s, v, T) - u(s', v, T)| \leq |s - s'| \tag{4.25}$$

u is continuous and convex in s for $s \in (0, \infty)$ and $s \mapsto u(s, v, T) - g(s)$ is non-decreasing in $[0, K]$ by Proposition 4.8 and Lemma 4.18. Without the loss of generality, it is sufficient to show (4.25) for these cases:

(i) $s < s' \leq b(v)$,

(ii) $s \leq b(v) < s'$,

(iii) $b(v) < s < s'$.

In all of these cases, $|u(s, v, T) - u(s', v, T)| = u(s, v, T) - u(s', v, T)$ since $u(s, v, T)$ is decreasing in s by Lemma 4.17.

In case (i),

$$u(s, v, T) - u(s', v, T) = (K - s) - (K - s') = |s - s'|$$

In case (ii), since $u(s', v, T) \geq K - s'$,

$$\begin{aligned} u(s, v, T) - u(s', v, T) &= (K - s) - u(s', v, T) \\ &\leq (K - s) - (K - s') \\ &= s' - s = |s - s'| \end{aligned}$$

In case (iii),

$$\begin{aligned} u(s, v, T) - u(s', v, T) &\leq u(0, v, T) - u(s' - s, v, T) \\ &\leq K - (K - (s - s')) = |s - s'| \end{aligned}$$

where the first inequality is justified by the fact $u(s, v, T)$ is convex and decreasing. The second inequality holds because $u(0, v, T) = K$ and $u(s' - s, v, T) \geq K - (s - s')$. By (4.25) and Lemma 4.18, we have that

$$\begin{aligned} u(s, v, T) - u(s', v', T) &= u(s, v, T) - u(s', v, T) + u(s', v, T) - u(s', v', T) \\ &\leq |s - s'| + C(|v - v'| + |v - v'|^{\frac{1}{2}}) \\ &\leq C(|s - s'| + |v - v'| + |v - v'|^{\frac{1}{2}}) \end{aligned}$$

□

Corollary 4.26. *$\tilde{u}(x, v)$ and $u(s, v)$ are locally $\frac{1}{2}$-Hölder continuous.*

4.4.2 Variational inequality and viscosity solution property of the value function

We recall the concept of viscosity solution described in Chapter 1 and adapt this for the BNS model. The aim of this subsection is to show the value function of the American put problem is the viscosity solution of a variational inequality.

We have already shown that (S, V) is a strong Markov process with state space $\mathcal{O} \stackrel{\text{def}}{=} (0, \infty) \times (0, \infty)$. Recall its infinitesimal generator acting on $C^{2,1}(\mathcal{O})$ is given by

$$\begin{aligned} Lf(s,v) = rs\partial_1 f(s,v) + \frac{1}{2}s^2 v\partial_2 f(s,v) - \lambda v \partial_2 f(s,v) \\ + \lambda \int_0^\infty f(se^{-\rho z}, v+z) - f(s,v) + s(1-e^{-\rho z})\partial_1 f(s,v) \mathit{\Pi}(\mathrm{d}z). \end{aligned}$$

The value function of the perpetual American put is a function of s and v only. We now define a notion of viscosity solution for the variational inequality relevant to the perpetual American put problem.

Let $\mathcal{W}$ be the set of function $f : \mathcal{O} \to \mathbb{R}$ that satisfy linear growth condition, i.e. there exists some C such that

$$|f(s,v) - f(s',v')| \leq C(1 + |s-s'| + |v-v'|)$$

Definition 4.27. *Consider the integral-partial differential equation (IPDE)*

$$\min(-Lf(s,v) + rf(s,v),\ f(s,v) - g(s)) = 0 \tag{4.26}$$

A function $h \in C^0(\bar{\mathcal{O}}) \cap \mathcal{W}$ is a viscosity subsolution (supersolution) of (4.26) if for all $(s,v) \in \mathcal{O}$ and for all $\psi \in C^{2,1}(\mathcal{O}) \cap \mathcal{W}$ such that

(i) $\psi(s,v) = h(s,v)$ and

(ii) for all $(s',v') \in \mathcal{O}$, $\psi(s',v') \geq h(s',v')$ ($\leq$), then

$$\min(-L\psi(s,v) + r\psi(s,v),\ \psi(s,v) - g(s)) \geq 0\ (\leq)$$

The function h is a viscosity solution if it is both a subsolution and a supersolution.

We define the parabolic superjets as

$$\mathcal{P}^{2,+}u(s,v) = \bigg\{(p,q,A) \in \mathbb{R}^3 : u(s',v') - u(s,v) \le p(v-v') + q(s-s') \\ + \frac{1}{2}A(s-s')^2 + o\bigg(|v-v'| + |s-s'|^2\bigg) \text{ as } (s'v') \to (s,v)\bigg\} \tag{4.27}$$

The definition given by (4.27) is different to the definition of semijets given Chapter 1. We have taken the parabolic definition of the superjets and subjets by treating v like a time component. We refer to [19] Chapter 8 for more information on parabolic formulation of viscosity solutions. This means we do not need to consider the second derivative with respect to time. We will discuss the reasoning behind this choice in Remark 4.33. The subjets are defined by

$$\mathcal{P}^{2,-}u(s,v) = -\mathcal{P}^{2,+}(-u)(s,v)$$

Proposition 4.28. *u(s,v) is a viscosity solution of equation (4.26).*

The proof of the viscosity solution property of the value function is very similar to the proof in [61] for the finite horizon case. It is found on page 135 in the chapter appendix.

Remark 4.29. One of the boundary conditions given by the author of [61] in Definition 3.1 is that $u(s,0,T) = g(s)$ for $T > 0$.

We show why this is incorrect, by showing that $u(s,0,t) > g(s) = 0$ for $s \ge K$. Recall Remark 4.2 (i), the process V has the representation

$$V_t^0 = e^{-\lambda t}\bigg(\int_0^t e^{\lambda q}\mathrm{d}Z_{\lambda q}\bigg).$$

so $P(V_t > 0) > 0$ for $t \in (0,T)$. By the supermartingale property of the value function,

$$\begin{aligned} u(s,0,T) &\ge e^{-rt}\mathbb{E}u(S_t^{s,v}, V_t^v, T-t) \\ &\ge e^{-rt}\mathbb{E}u(S_t^{s,v}, V_t^v, T-t)\mathbf{1}_{\{V_t^v>0\}}, \end{aligned} \tag{4.28}$$

where the second inequality follows by the positivity of u. We consider the cases, $S_t < K$ and $S_t \ge K$.

(i) If $S_t^{s,v} < K$, then $u(S_t^{s,v}, V_t^v, T-t) \ge g(S_t^{s,v}) = K - S_t^{s,v} > 0$.

(ii) If $S_t^{s,v} \ge K$, then $S_t^{s,v} > b(V_t, T-t)$ by Lemma 4.19, so $u(S_t^{s,v}, V_t^v, T-t) > g(S_t^{s,v}) = 0$.

It follows that (4.28) must be strictly positive, because it is the expectation of a strictly positive function over a set with strictly positive measure.

This finding is similar to the observation in [3, Remark 4] for the Bates model (also known as the Heston model with jumps), which is also a stochastic volatility model with jumps. In the case of the Bates model, the variance process cannot escape from 0, but the process S has a jump part independent of the variance process, so the value of the option is non-zero even if the variance process starts in 0. In the finite element scheme used in [3], no boundary condition is prescribed at $v = 0$.

We have shown that the value function of the optimal stopping problem is a solution of the variational inequality but we have not shown that it is unique. For the purpose of showing the value function is smooth in the continuation region, uniqueness is not required. We shall see this in the next subsection. It is unclear to us, without the boundary condition at $v = 0$, (in both finite horizon and infinite horizon) whether the solution to (4.26) is unique.

4.4.3 $C^{2,1}$ smoothness for the value function in the continuation region for the infinite horizon problem

The result in this section utilises the result from the previous section. We have established in Chapter 1 that any classical solution constitutes a viscosity solution, but not vice versa.

In this section, under the additional assumption that the Lévy is finite, we demonstrate that if we restrict value function to the continuation region D^c, then the value function satisfies the equation

$$-Lf(s,v) + rf(s,v) = 0 \tag{4.29}$$

in the classical sense. This means for every point in the interior of the continuation region, the first and second derivative with respect to s and the first derivative with respect to v exist. Moreover, these derivatives are locally Hölder continuous. Smoothness of the value function of the American put is of interest to practitioners as it helps justify the use of PDE discretisation schemes such as the finite element used to solve for the IPDE (4.26) numerically.

Assumption 4.30. *We assume that the Lévy measure* Π *is a finite measure. That is to say*

$$\Pi(0,\infty) < \infty$$

In order to prove that the value function $u(s,v)$ is a classical solution, we need the following auxiliary lemma about the operator

$$\tilde{L} = \frac{1}{2}s^2 v\partial_{11}f(s,v) + (r + C_\rho)\partial_1 f(s,v) - \lambda v\partial_2 f(s,v) - rf(s,v) \tag{4.30}$$

Lemma 4.31. *Consider the Cauchy-Dirichlet problem*

$$
\begin{cases}
-\tilde{L}f(s,v) + \gamma(s,v) = 0 & \text{for } (s,v) \in (s_1,s_2) \times (v_1,v_2), \\
f(s,v_1) = \zeta(s) & \text{for } s \in [s_1,s_2] \\
f(s,v) = \psi(s,v) & \text{for } (s,v) \in \{s_1,s_2\} \times [v_1,v_2).
\end{cases}
\tag{4.31}
$$

where γ, g and ψ are continuous. If there is a viscosity solution f_1 and a classical solution f_2 to the problem (4.31), then $f_1 = f_2$.

The proof of this lemma is an application of maximum principle given [19, Theorem 8.2]. See page 137 of the chapter appendix for a proof.

Proposition 4.32. *The value function $u(s,v)$ satisfies (4.29) in the classical sense in the continuation D^c. Moreover, the derivatives*

$$\partial u_s(s,v),\ \partial_{ss}u(s,v),\ \partial_v u(s,v)$$

exist and they are $\frac{1}{2}$-Hölder continuous for $(s,v) \in D^c$.

Proof. On a domain $\Theta = (s_1,s_2) \times (v_1,v_2) \subset D^c$, consider the Cauchy-Dirichlet problem (as an operator acting on f)

$$
\begin{aligned}
-\lambda \int_0^\infty & u(se^{-\rho z}, v{+}z) - u(s,v) + s(1-e^{-\rho z})\partial_1 f(s,v)\Pi(\mathrm{d}z) \\
& -rs\partial_1 f(s,v) - \frac{1}{2}s^2 v\partial_{11} f(s,v) + \lambda v \partial_2 f(s,v) + rf(s,v) = 0 \quad \text{for } (s,v) \in \Theta,
\end{aligned}
\tag{4.32}
$$

$$f(s,v) = u(s,v) \quad \text{ on } [s_1,s_2] \times \{v_1\} \cup \{s_1,s_2\} \times (v_1,v_2). \tag{4.33}$$

Here u is the value function of the BNS perpetual American put and is treated as a known function for the purpose of this proof. Define

$$\gamma(s,v) = -\lambda \int_0^\infty u(se^{-\rho z}, v+z) - u(s,v)\Pi(\mathrm{d}z)$$

and

$$C_\rho = \lambda \int_0^\infty (1-e^{-\rho z})\Pi(\mathrm{d}z),$$

For all $\rho \geq 0$, there exists c such that for $z \in (0,c)$

$$1 - e^{-\rho z} \leq cz. \tag{4.34}$$

By (4.1) we see that $C_\rho < \infty$. Moreover, under Assumption 4.30, γ has the same property we proved for u in Proposition 4.25

$$\begin{aligned}|\gamma(s,v)-\gamma(s',v')| &= \lambda\left|\int_0^\infty u(se^{-\rho z},v+z)-u(s,v)-u(s'e^{-\rho z},v'+z)+u(s',v')\Pi(\mathrm{d}z)\right|\\ &\leq \lambda\int_0^\infty \left|u(se^{-\rho z},v+z)-u(s'e^{-\rho z},v'+z)\right|+\left|u(s',v')-u(s,v)\right|\Pi(\mathrm{d}z)\\ &\leq \lambda C\int_0^\infty |s-s'|(1+e^{-\rho z})+2|v-v'|+2|v-v'|^{\frac{1}{2}}\Pi(\mathrm{d}z)\\ &\leq 2\lambda C\Pi(0,\infty)(|s-s'|+|v-v'|+|v-v'|^{\frac{1}{2}}),\end{aligned}$$

where the last line follows the penultimate line by Proposition 4.25. This means γ is a locally $\frac{1}{2}$-Hölder continuous function.

Equation (4.32) can be rewritten as

$$-\frac{1}{2}s^2v\partial_{11}f(s,v)-(r+C_\rho)s\partial_1 f(s,v)+\lambda v\partial_2 f(s,v)+rf(s,v)+\gamma(s,v)=0 \quad \text{for } (s,v)\in\Theta. \tag{4.35}$$

Note equation (4.35) is uniformly parabolic with smooth coefficients and γ is a locally $\frac{1}{2}$-Hölder continuous function on Θ. By Corollary 2 on page 71 of [28], the pair of equations (4.33) and (4.35) have a unique solution $\bar{u}$ with $\bar{u}\in C^{2,1}(\Theta)$. The derivatives $\partial_{ss}\bar{u},\partial_s\bar{u},\partial_v\bar{u}$ are locally $\frac{1}{2}$-Hölder continuous.

We observe that $u(s,v)|_\Theta$ is a viscosity solution to equations (4.33) and (4.35). By Lemma 4.31, we have that $\bar{u}(s,v)=u|_\Theta(s,v)$. Since the continuation region D^c is an open set, for every $(s,v)\in D^c$, it is possible to find s_1,s_2,v_1,v_2 such that $(s,v)\in(s_1,s_2)\times(v_1,v_2)\subset D^c$ hence u is $C^{2,1}$ for every $(s,v)\in D^c$. □

Remark 4.33. We make the following observations about the previous proposition.

(i) In the proof of Proposition 4.32, Assumption 4.30 is only required to show that the integral related term γ is Hölder continuous. Moreover, we did not use the fact that $g(\cdot)=(K-\cdot)^+$. Proposition 4.32 applies to the continuation region of optimal stopping problems under the BNS model if we can show $u(s,v)$ and $\gamma(s,v)$ are locally Hölder continuous when restricted to the continuation region.

(ii) We did not use the fact that the value function u is the unique solution to (4.26). If fact we do not know if this holds, which we have discussed in Remark 4.29. Instead, we used uniqueness of solution to equations (4.33) and (4.35) on $\Theta\subset D^c$. Since for every $(s,v)\in D^c$, we can find such Θ, $u(s,v)|_{D^c}\in C^{2,1}(D^c)$.

(iii) We identified the unique $C^{2,1}$ solution to (4.33) and (4.35) with $u(s,v)$ by Lemma 4.31. In the proof of this auxiliary lemma, we need to use the superjet definition of

(4.27) and its subjet counterpart in order to appeal to the maximum principle for parabolic problems.

The problem with using the general degenerate elliptic definition is that the classical comparison principle (Theorem 1.10 on page 14) requires two candidate solutions to agree on the whole of $\partial\Theta$. On the other hand, the results in [28] for parabolic problems does not impose a boundary condition for $(s, v) \in [s_1, s_2] \times \{v_2\}$. This is because the variable v takes the role of time in the standard parabolic set-up and the first initial boundary problems do not allow the imposition of a terminal condition.

(iv) For any $(s, v) \in D^c$, if we can find an open set Θ of the form $\Theta = (s_1, s_2) \times (v_1, v_2)$ such that $\gamma(s, v)|_\Theta \in C^{2,1}(\Theta)$, then it is possible to iterate the theorem in [28] to show that $u(s, v)|_\Theta \in C^{4,2}(\Theta)$. The arguments can be repeated to show that the value function has higher order smoothness in certain parts of the continuation region.

For a given (s, v), γ is $C^{2,1}$ at point (s, v) if both conditions given below are satisfied.

(1) For all z in the support of the measure Π,

$$(se^{-\rho z}, v + z) \neq (b(v_0), v_0),$$

for any $v_0 \in (0, \infty)$.

(2) The measure Π satisfies the correct condition such that the derivative is finite.

To illustrate these conditions, consider the derivative of γ with respect to v. By condition (1), $u(se^{-\rho z}, v + z)$ is differentiable with respect to v in the support of z. So,

$$\begin{aligned}
\frac{\partial\gamma(s, v)}{\partial v} &= \int_0^\infty \frac{\partial u}{\partial v}(se^{-\rho z}, v + z) - \frac{\partial u}{\partial v}(s, v)\Pi(\mathrm{d}z) \\
&\leq \int_0^\infty |se^{-\rho z} - s|^{\frac{1}{2}} + |v + z - v|^{\frac{1}{2}}\Pi(\mathrm{d}z) \\
&\leq \int_0^\infty (2^{\frac{1}{2}} s^{\frac{1}{2}} + z^{\frac{1}{2}})\Pi(\mathrm{dz}),
\end{aligned}$$

where the first inequality follows because u is $\frac{1}{2}$-Hölder. In this case, a sufficient condition for the γ to be differentiable with respect to v is that

$$\int_0^\infty z^{\frac{1}{2}}\Pi(\mathrm{d}z) < \infty \tag{4.36}$$

For a particular (s, v), if there exists v_0 such that

$$(se^{-\rho z}, v + z) = (b(v_0), v_0),$$

and z is within the support of Π, then it is not clear to us whether the u has better regularity condition than $C^{2,1}$ at (s,v). We illustrate that it is possible for the process (S,V) to jump into the stopping region from the continuation region.

Lemma 4.34. *For all λ, ρ and measure Π with full support on $(0,\infty)$, there exists r such that it is possible for the process (S_t,V_t) to jump from the continuation region D^c to the stopping region D.*

Proof. For the purpose of this proof only, we write $u(s,v;r)$ to denote the value function and $b(v;r)$ to emphasise the dependence of u on the parameter r.

We shall prove that there exists some $r>0$ such that $b(v;r)>Ke^{-\rho v}$. Since $(K,0)$ is in the continuation region for any $r>0$ and $(Ke^{-\rho v},v)$ is in the stopping region; it is possible to jump into the stopping region.

We assume that $b(v;r)\le Ke^{-\rho v}$, then for a fixed (s,v) such that $Ke^{-\rho v}<s<K$, this point is in the continuation region for all $r>0$. By the martingale property of u up to an optimal stopping time, we have that

$$u(s,v;r)=\mathbb{E}e^{-rT_1}u(S^{s,i}_{T_1},V^{s,i}_{T_1})\mathbf{1}_{\{\tau>T_1\}}+\mathbb{E}e^{-r\tau}g(S^{s,i}_{\tau})\mathbf{1}_{\{\tau\le T_1\}},\tag{4.37}$$

where T_1 the time of first jump and $\tau=\inf\{t:S^{s,v}_t\le b(ve^{-\lambda t};r)\}$.

T_1 has an exponential distribution with mean $\mu=(\lambda\Pi(0,\infty))^{-1}$. We estimate the terms on the right hand side of equation (4.37) separately. For the first term, since u is bounded above by K, we have that

$$\mathbb{E}e^{-rT_1}u(S^{s,i}_{T_1},V^{s,i}_{T_1})\mathbf{1}_{\{\tau>T_1\}}\le\mathbb{E}e^{-rT_1}K\le\frac{K}{1+r\mu^{-1}}.$$

For the second term, on the set $\{\tau\le T_1\}$, we have

$$S_t=s\exp\bigg(\int_0^t\sqrt{ve^{-\lambda t}}\mathrm{d}W_t-\frac{1}{2}\int_0^t ve^{-\lambda t}\mathrm{d}t+(r+\lambda\Phi(\rho))t\bigg).$$

Now consider the process $\hat{S}^r$, indexed by r, defined by

$$\hat{S}^r_t=s\exp\bigg(\int_0^t\sqrt{ve^{-\lambda t}}\mathrm{d}W_t+(r-v)t\bigg)\text{ for }t\ge 0.$$

Since $S_t\ge\hat{S}^r_t$, we have that

$$\mathbb{E}e^{-r\tau}g(S^{s,i}_{\tau})\mathbf{1}_{\{\tau>T_1\}}\le\mathbb{E}e^{-r\tau_1}g(\hat{S}^r_{\tau_1})\le K\mathbb{P}(\tau^r_1<\infty)\le K\mathbb{P}(\tau^r_2<\infty),$$

where $\tau_1=\inf\{t:S^r_t\le b(ve^{-\lambda t})\}$ and $\tau_2=\inf\{t:S^r_t\le Ke^{-\rho ve^{-\lambda t}}\}$. The first inequality hold because g is a positive decreasing function and the drift of $\hat{S}^r$ is smaller than the drift of S. The second inequality hold because g is bounded by K and $e^{-r\tau} < 1$. The third inequality holds because we assumed that $b(ve_t^\lambda) < Ke^{-\rho ve^{-\lambda t}}$ and $s > b(v)$.

By (4.37), we have that

$$K - s < u(s, v; r) \leq \frac{K}{1 + r\mu^{-1}} + K\mathbb{P}(\tau_2^r < \infty) \to 0 \quad \text{as } r \to \infty,$$

which is a contradiction. □

4.4.4 Smooth pasting condition in the infinite horizon problem

Let us denote the boundary of the continuation region by ∂D^c. So far, we have used the phrase **stopping boundary** to mean $b(v)$ in the infinite horizon case. If b is continuous, then $b(v)$ and ∂D^c are related in the following way:

$$\partial D^c = \{(v, b(v)) : v \in [0, \infty)\} \cup \{\{0\} \times (b(0), \infty)\}.$$

We have shown that b is left-continuous in Proposition 4.22. Since b is a decreasing function, we know left and right limits exist at every point. Let $\beta(v)$ to be the right limit of $b(v)$ at v; that is to say,

$$\beta(v) = \lim_{v^* \downarrow v} b(v^*).$$

If b is discontinuous at some $\tilde{v}$, then

$$\{\{\tilde{v}\} \times (\beta(\tilde{v}), b(\tilde{v}))\} \subset \partial D^c.$$

The goal of this subsection is to show $\partial_1 u(s, v)$ is continuous across the stopping boundary $b(v)$. This means $\partial_1 u(s, v)$ is continuous everywhere in $\mathcal{O}$. In order to do this, we first show that partial derivatives, which are well defined in the interior of the continuation region D^c, can be extended continuously to ∂D^c. We now show that the smooth pasting condition holds.

Proposition 4.35. *u exhibits smooth pasting across the stopping boundary in s. This is to say, for all $v \in (0, \infty)$*

$$\lim_{D^c \ni (s', v') \to (s, v)} \frac{\partial u}{\partial s}(s, v') = -1 \quad \textit{for } s \in [\beta(v), b(v)].$$

Proof. This proof is very similar to the proof given by Zhang in [77] for jump diffusion model in finite time horizon. By Corollary 4.21, we see that

$$-1 \leq \partial_1 u(s, v) \leq 0, \tag{4.38}$$

The variational inequality tells us that u satisfies the following equation in the viscosity (or distribution) sense:

$$\begin{aligned}\frac{1}{2}s^2 v\partial_{11}u(s,v) \leq & ru(s,v) \underbrace{-rs\partial_1 u(s,v)}_{\leq rs \text{ by (4.38)}} + v\partial_2 u(s,v) \\ & - \int_0^\infty u(se^{-\rho z}, v+z) - u(s,v) + s(1-e^{-\rho z})\partial_1 u(s,v)\Pi(\mathrm{d}z) \qquad (4.39)\end{aligned}$$

Note that

$$u(se^{-\rho z}, v+z) \geq u(se^{-\rho z}, v) \geq u(s,v) + s(e^{-\rho z}-1)\partial_1 u(s,v), \qquad (4.40)$$

where the first inequality holds because $u(s,\cdot)$ is an increasing function. The second inequality holds by convexity of $u(\cdot, v)$. This means the integral part of the equation (4.39) is non-negative, hence

$$\begin{aligned}\frac{1}{2}s^2 v\partial_{11}u(s,v) &\leq ru(s,v) - rs\partial_1 u(s,v) + \lambda v\partial_2 u(s,v) \\ &\leq ru(s,v) + rs + \lambda v\partial_2 u(s,v).\end{aligned}$$

Since $\partial_2 u(s,v)$ and $\partial_{11}u(s,v)$ are both locally bounded, by Chapter 2, Lemma 3.1 of [43], we may conclude that $\partial_1 u(s,v)$ is continuous across the stopping boundary. □

Remark 4.36. Smooth pasting condition is often assumed in numerical IPDE schemes, for example see [3].

4.4.5 Continuity of stopping boundary for the infinite horizon problem

In this section, we prove the stopping boundary $b(v)$ is continuous. $b(v)$ is also referred to as the free-boundary of (4.26) because the equation

$$-Lu + ru = 0$$

is satisfied for values above the unknown boundary $b(v)$, i.e.

$$\{(s,v) : s > b(v)\}.$$

Properties of free boundary for parabolic variational inequalities have been studied by various authors, see for example, see [27], [35] and [74]. In [58], Pham studied the jump diffusion model in finite horizon, which also has a parabolic differential integral operator. The approach we take in proving the continuity of the stopping boundary is similar to the approach in [35] and [58], but the difference is that we do not have a time component.

We first show the second derivative of the value function is bounded away from 0 near the stopping boundary.

Lemma 4.37. *The second derivative of u with respect to s satisfies the inequality*

$$\lim_{D^c \ni (s',v') \to (s,v)} \frac{1}{2} s'^2 v' \partial_{11} u(s',v') \geq K\left(r - \lambda \int_0^\infty e^{-\rho z} \Pi(dz)\right) \quad \textit{for } s \in [\beta(v), b(v)].$$

Proof. By Proposition 4.32, we have that

$$\begin{aligned}
&\lim_{D^c \ni (s',v') \to (s,v)} \frac{1}{2} s'^2 v' \partial_{11} u(s',v') \\
&\qquad = \lim_{D^c \ni (s',v') \to (s,v)} r \underbrace{u(s',v')}_{\geq K - s'} - rs' \underbrace{\partial_1 u(s',v')}_{\to -1 \text{ by Prop.4.35}} + \lambda \underbrace{v' \partial_2 u(s',v')}_{\geq 0 \text{ by Prop. 4.8}} \\
&\qquad - \lim_{D^c \ni (s',v') \to (s,v)} \lambda \int_0^\infty u(s' e^{-\rho z}, v' + z) - u(s' e^{-\rho z}, v') \Pi(\mathrm{d}z) \\
&\qquad - \lim_{D^c \ni (s',v') \to (s,v)} \lambda \int_0^\infty \big(u(s' e^{-\rho z}, v') - u(s',v') + s'(1 - e^{-\rho z}) \partial_1 u(s',v')\big) \Pi(\mathrm{d}z)
\end{aligned}$$

We estimate each of the integral terms. The first integral term can be estimated by

$$\begin{aligned}
&\lim_{D^c \ni (s',v') \to (s,v)} \int_0^\infty u(s' e^{-\rho z}, v' + z) - u(s' e^{-\rho z}, v') \Pi(\mathrm{d}z) \\
&\qquad \leq \int_0^\infty K - (K - s e^{-\rho z}) \Pi(\mathrm{d}z) = s \int_0^\infty e^{-\rho z} \Pi(\mathrm{d}z),
\end{aligned}$$

where we used $u(s' e^{-\rho z}, v' + z) \leq K$ and the continuity of u. The integrand of the second integral can be shown to converge to 0.

$$\begin{aligned}
&\lim_{D^c \ni (s',v') \to (s,v)} u(s e^{-\rho z}, v) - u(s,v) + s(1 - e^{-\rho z}) \underbrace{\partial_1 u(s,v)}_{\to -1 \text{ by Prop. 4.35}} \\
&\qquad = u(s e^{-\rho z}, v) - u(s,v) + s(1 - e^{-\rho z})(-1) \\
&\qquad = K - s e^{-\rho z} - (K - s) + s(1 - e^{-\rho z})(-1) = 0,
\end{aligned}$$

where we used continuity of u for the first inequality. The second equality follows from $u(s,v) = (K - s)$ for $s \in (0, b(v)]$. Hence

$$\begin{aligned}
\lim_{D^c \ni (s',v') \to (s,v)} \frac{1}{2} s'^2 v' \partial_{11} u(s',v') &\geq r(K - s) + rs - s \int_0^\infty e^{-\rho z} \Pi(\mathrm{d}z) \\
&\geq K\left(r - \lambda \int_0^\infty e^{-\rho z} \Pi(\mathrm{d}z)\right)
\end{aligned}$$

□

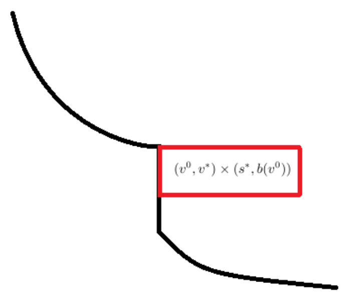
This lemma allows us to prove that $b(v)$ is continuous. The approach used is based on the Newton-Leibniz formula.

Proposition 4.38. *If* r, λ, Π *satisfies the condition*

$$C = r - \lambda \int_0^\infty e^{-\rho z} \Pi(dz) > 0,$$

then the function $b(v)$ *is continuous for* $v \in [0, \infty)$*,*

Proof. Assume that $b(v)$ is discontinuous at some point v^0. By Lemma 4.37, there exists s^*, v^* such that $(s, v) \in \frac{1}{2}s^2 v \partial_{11} u(s, v) > C/2$ for $(s, v) \in (s^*, b(v^0)) \times (v^0, v^*)$. This is illustrated in the diagram below.



There exists $C_1 > 0$ such that

$$\partial_{11} u(s, v) \geq C_1 \quad \text{for } (s, v) \in (v^0, v^*) \times (s^*, b(v^0)). \tag{4.41}$$

By linearity of g on $(0, K)$, for $s > 0$, we have that

$$g(b(v^0)) = K - b(v^0) = (K - s) + \int_s^{b(v^0)} (-1) \mathrm{d}y. \tag{4.42}$$

Moreover, take a sequence $(v_n, n \geq 1)$ converging to v^0 such that $v_n \in (v^0, v^*)$ for every n. Then for any $s \in (s^*, b(v^0))$, the value function admits the following representations:

$$\begin{aligned} u(b(v^0), v_n) &= u(s, v_n) + \int_s^{b(v^0)} \partial_1 u(y, v_n) \mathrm{d}y \\ &= u(s, v_n) + \int_s^{b(v^0)} \left(\partial_1 u(s, v_n) + \int_s^y \partial_{11} u(z, v_n) \mathrm{d}z \right) \mathrm{d}y \end{aligned} \tag{4.43}$$

Combining the expression (4.42) and (4.43), we have that

$$
\begin{aligned}
&u(b(v^0), v_n) - g(b(v^0)) \\
&\quad = \underbrace{u(s, v_n) - g(s)}_{\geq 0 \text{ by definition of } u} + \underbrace{\int_s^{b(v^0)} (\partial_1 u(s, v_n) + 1)\mathrm{d}y}_{\geq 0 \text{ by Corollary 4.21}} + \int_s^{b(v^0)} \int_s^y \underbrace{\partial_{11} u(s, v_n)}_{\geq C_1 \text{ by (4.41)}} \,\mathrm{d}z\mathrm{d}y \\
&\quad \geq \int_s^{b(v^0)} \int_s^y C_1 \mathrm{d}z\mathrm{d}y \\
&\quad = \frac{C_1}{2}(b(v_0) - s)^2.
\end{aligned}
$$

We now let $v_n \downarrow v^0$, the left hand side converges to 0, which is a contradiction. □

4.5 Regularity property of value function and stopping boundary for the finite horizon American put problem

In the perpetual American put problem, we have shown the value function is $C^{2,1}$ except on the stopping boundary. This is due to the fact, the operator $L - r$ is parabolic. By treating the integral part of the operator $L - r$ as a forcing term, we can use parabolic PDE theory to show the value function in the continuation region is $C^{2,1}$.

In the finite horizon case, the value function satisfies a variational equation analogous to (4.26), except it has an additional term to take account of the time horizon, i.e.

$$\min((\partial_t - L + r)u(s, v, t), u(s, v, t) - g(s)) = 0 \tag{4.44}$$

with an extra boundary condition $u(s, v, 0) = g(s)$.

Ideally, the method used for showing differentiability property of the value function in the perpetual case can be repeated for the finite horizon problem. The most natural generalisation of a parabolic operator is a hypoelliptic operator. If we can show that $-\partial_t + L - r$ is a "well-behaved" operator like $L - r$, then we can appeal to theorems similar to the one below.

Theorem 4.39. *Let $\mathcal{L}$ denote the operator*

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{k} X_i^2 + X_0, \tag{4.45}$$

where $X_0, X_1, \ldots, X_k$ are C^∞ vector fields in $\mathbb{R}^n$. Assume that Lie Algebra generated by $X_0, X_1, \ldots, X_k$ spans $\mathbb{R}^n$. Then, if u is a distribution such that

$$\mathcal{L}u = f$$

and $f \in C^\infty$ in an open set G, then u is C^∞ in G.

This theorem first appeared in [34] and the Lie Bracket condition associated with this theorem is known as Hörmander's condition. A jump related integral similar to γ in Proposition 4.32 would play the role of f. We know f is not C^∞, but there are many related results. These results only assumes f to be in a Sobelev space $W^{m,p}$, in which case $u \in W^{m+2-\epsilon,p}$ for some $\epsilon \geq 0$. The value of ϵ depends on the minimum number of Lie Brackets needed to span $\mathbb{R}^n$. See, for example [62, Theorem 18].

Unfortunately, the differential part of the operator $-\partial_t + L - r$ does not satisfy Hörmander's condition, hence we cannot appeal to these results. The non-integral part of $-\partial_t + L - r$ is $-\partial_t + \frac{1}{2}s^2 v\partial_{ss} + (r + C_\rho)s\partial_s - \lambda v\partial_v - (r + C_1)$, where $C_1 = \lambda\Pi(0,\infty)$. Following notation of (4.45), then

$$\mathcal{L} = -\partial_t + \frac{1}{2}s^2 v\partial_{ss} + (r + C_\rho)s\partial_s - \lambda v\partial_v - (r + C_1), \tag{4.46}$$

where X_0 and X_1 can be defined as follows:

$$X_0 = (C_\rho + r - \frac{1}{2}v)s\partial_s - \lambda v\partial_v - \partial_t - (r + C_1) \quad \text{and} \quad X_1 = s\sqrt{v}\partial_s.$$

A straightforward calculation shows that

$$[X_0, X_1] = 0.$$

This means the Hörmander's condition does not hold for the truncated PIDE operator. On the process level, this corresponds to the fact, when the jumps are ignored, the process v is a deterministic exponential decaying process. It is therefore unclear to us whether $u(s,v,t)|_{D^c} \in C^{2,1,1}(D^c)$ still holds, but it is still possible to prove some regularity properties of the value function.

4.5.1 Hölder continuity of the value function

Proposition 4.40. *For $T, T' < \infty$, the value function u satisfies*

$$\begin{aligned}&|u(s,v,T) - u(s',v',T')| \\ &\quad \leq C_0(|s - s'| + |v - v'| + |v - v'|^{1/2}) + C'(1 + v^{\frac{1}{2}} + v)(|T - T'|^{1/2} + |T - T'|),\end{aligned}$$

where C_0 and C' are constants.

Proof. Recall Proposition 4.25, which states

$$|u(s,v,T') - u(s',v',T')| \leq C_0(|s - s| + |v - v'|^{1/2} + |v - v'|). \tag{4.47}$$

By triangle inequality, we have

$$|u(s,v,T)-u(s',v',T')| \le |u(s,v,T)-u(s,v,T')| + |u(s,v,T')-u(s,'v',T')|, \tag{4.48}$$

it is therefore only necessary to estimate $|u(s,v,T)-u(s,v,T')|$. Without loss of generality, we assume that $T' \ge T$. By the definition of u, we have $u(s,v,T') \ge u(s,v,T)$ for $T' \ge T$. For a fixed $\delta = T' - T$, define a deterministic process $\tilde{r}(t)$ such that

$$\tilde{r}(t) = \begin{cases} 0 & \text{for } 0 \le t < \delta \\ r & \text{for } \delta \le t \le T'. \end{cases} \tag{4.49}$$

Then $\tilde{r}(t) \le r$ for all $t \ge 0$. Now consider the optimal stopping problem

$$w_{T,\delta}(s,v,t) = \sup_{t \le \tau \le T'} \mathbb{E}[e^{-\int_0^\tau \tilde{r}(t+q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(t+q)-r)\mathrm{d}q} S_\tau^{s,v})] \quad \text{for } 0 \le t \le T'.$$

We make the following observations about w.

(i) $w_{T,\delta}(s,v,t) \ge u(s,v,T'-t)$ for all $t \in [0,T]$. This follows from

$$e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i}) \ge e^{-r\tau} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i}) \ge e^{-r\tau} g(S_\tau^{s,i}),$$

where the first inequality holds because $\tilde{r}(t) \le r$. The second inequality holds because g is decreasing and $e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} < 1$.

(ii) $w_{T,\delta}(s,v,t) = u(s,v,T'-t)$ for $\delta \le t \le T'$.

(iii) It is not optimal to stop before δ. First observe that

$$e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i}) = (K - e^{-rt} S_t)^+ \quad \text{for } 0 \le t \le \delta. \tag{4.50}$$

Since $e^{-rt}S_t$ is a martingale, $(K - e^{-rt}S_t^{s,i})^+$ is a submartingale by convexity $(K-\cdot)^+$. Then, for any stopping time τ,

$$\begin{aligned} \mathbb{E}[(K-e^{-r\delta}S_\delta)^+ \mathbf{1}_{\{\tau \le \delta\}}] &= \mathbb{E}[\mathbb{E}[(K-e^{-r\delta}S_\delta)^+ \mathbf{1}_{\{\tau \le \delta\}} | \mathcal{F}_\tau]] \\ &\ge \mathbb{E}[\mathbf{1}_{\{\tau \le \delta\}} (K - \mathbb{E}[e^{-r\delta}S_\delta | \mathcal{F}_\tau])^+] \\ &= \mathbb{E}[\mathbf{1}_{\{\tau \le \delta\}} (K - e^{-r\tau} S_\tau)^+], \end{aligned}$$

where the inequality holds by Jensen's inequality for conditional expectation and the last equality holds by optional sampling theorem as τ is a bounded stopping time.

Hence, for any stopping time $\tau \leq T'$,

$$\begin{aligned}
&\mathbb{E}[e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i})] \\
&\quad = \mathbb{E}e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i})\mathbf{1}_{\{\tau\leq\delta\}} + \mathbb{E}e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i})\mathbf{1}_{\{\tau>\delta\}} \\
&\quad \leq \mathbb{E}e^{-\int_0^\delta \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\delta (\tilde{r}(q)-r)\mathrm{d}q} S_\delta^{s,i})\mathbf{1}_{\{\tau\leq\delta\}} + \mathbb{E}e^{-\int_0^\tau \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^\tau (\tilde{r}(q)-r)\mathrm{d}q} S_\tau^{s,i})\mathbf{1}_{\{\tau>\delta\}} \\
&\quad = \mathbb{E}[e^{-\int_0^{\tau\vee\delta} \tilde{r}(q)\mathrm{d}q} g(e^{\int_0^{\tau\vee\delta} (\tilde{r}(q)-r)\mathrm{d}q} S_{\tau\vee\delta}^{s,i})]
\end{aligned}$$

Let τ^* be an optimal stopping time at which the value function $w_{T,\delta}(s,v,t)$ is attained. Since the time horizon is bounded, τ^* exists. We must have $\tau^* \geq \delta$ by bullet point (iii). By martingale property of

$$\{e^{-\int_0^{t\wedge\tau^*} \tilde{r}(q)\mathrm{d}q} w_{T,\delta}(S_{t\wedge\tau^*}^{s,v}, V_{t\wedge\tau^*}^v, t\wedge\tau^*) : 0 \leq t \leq T'\},$$

we have

$$w_{T,\delta}(s,v,0) = \mathbb{E}u(S_\delta^{s,v}, V_\delta^v, T),$$

where we set $t = \delta$ and used bullet point (ii). Now by bullet point (i), we have

$$\begin{aligned}
&u(s,v,T') - u(s,v,T) \\
&\quad \leq w_{T,\delta}(s,v,0) - u(s,v,T) \\
&\quad = \mathbb{E}u(e^{-r\delta} S_\delta^{s,v,}, V_v, T) - u(s,v,T) \\
&\quad = \mathbb{E}\tilde{u}(X_\delta^{\log(s),v} - r\delta, V_t^v, T) - \tilde{u}(\log(s), v, T),
\end{aligned}$$

where $\tilde{u}(\log(s),v,T) = u(s,v,T)$. $\tilde{u}$ is the value function in term of the log price, which was first defined on page 98. By Proposition 4.24 and the explicit representation of X (4.5) on page 85, we have

$$\begin{aligned}
&u(s,v,T') - u(s,v,T) \\
\leq &C\mathbb{E}\left|X_\delta^{\log(s),v} - r\delta - x\right| + C\mathbb{E}|V_\delta^v - v| + C\mathbb{E}|V_\delta^v - v|^{\frac{1}{2}} \\
= &C\mathbb{E}\left|\int_0^\delta \sqrt{V_q^v}\mathrm{d}W_q - \frac{1}{2}\int_0^\delta V_q^v \mathrm{d}q + \lambda\Phi(\rho)\delta - \rho Z_{\lambda\delta}\right| + C\mathbb{E}|V_\delta^v - v| + C\mathbb{E}|V_\delta^v - v|^{\frac{1}{2}} \\
\leq &C\mathbb{E}\left|\int_0^\delta \sqrt{V_q^v}\mathrm{d}W_q\right| + C\mathbb{E}\left|\frac{1}{2}\int_0^\delta V_q^v \mathrm{d}q\right| + C\lambda\Phi(\rho)\delta + C\rho\mathbb{E}Z_{\lambda\delta} + C\mathbb{E}|V_\delta^v - v| + C\mathbb{E}|V_\delta^v - v|^{\frac{1}{2}},
\end{aligned}$$

From this, it is possible to show that

$$|u(s,v,T') - u(s,v,T)| \leq C'(1 + v^{\frac{1}{2}} + v)(|T - T'|^{\frac{1}{2}} + |T - T'|).$$

The detailed calculations are is found on page 136. This inequality completes the proof by (4.47) and (4.48). □

From Proposition 4.40, we can now conclude that $u(s,v,T)$ is locally Hölder continuous.

Corollary 4.41. *$u(s,v,T)$ is locally $\frac{1}{2}$-Hölder continuous.*

4.5.2 Differentiability properties of the value function

In this section, we prove that u is differentiable in two directions by transforming u and appealing to the same PDE result in the finite horizon. However, it is much more difficult to identify the candidate function with the value function.

Proposition 4.42. *Consider the transformation of variable $(s,v,t) \to (s,\hat{v},t)$, where $\hat{v} = ve^{-\lambda t}$. Define the function $\hat{u}(s,\hat{v},t) = u(s,\hat{v}e^{\lambda t},t)$, then the derivatives*

$$\partial_s \hat{u}, \partial_{ss}\hat{u}, \partial_t \hat{u}$$

exist in the continuity region and are $\frac{1}{2}$-Hölder continuous.

This shows u is twice differentiable with respect to s and differentiable in the direction $(0,\lambda v,1)$ inside the continuation region. Before we proceed with the proof of this result, we lay out the key steps of the proof in the following remark.

Remark 4.43. (i) Let f be a classical solution to the equation

$$-\mathcal{L}f + h = 0$$

where $\mathcal{L}$ is given by the equation (4.46) on a domain characterised by

$$\{(s,v,t) : (s,v,t) \in (s_1,s_2) \times (v_1 e^{\lambda t}, v_2 e^{\lambda t}) \times (t_1,t_2)\} \tag{4.51}$$

We make a transformation of variable. Let $(s,v,t) \to (s,\hat{v},t)$ be a transformation of variable where $\hat{v} = ve^{-\lambda t}$. Under this transformation, we have

$$-\partial_t \hat{f}(s,\hat{v},t) + \frac{1}{2}s^2 \hat{v}e^{\lambda t}\partial_{ss}\hat{f}(s,\hat{v},t) + (r - C_\rho)s\partial_s \hat{f}(s,\hat{v},t) - (r + C_1)\hat{f}(s,\hat{v},t) = \hat{h}. \tag{4.52}$$

on a cuboid of the form

$$\{(s,v,t) : (s,\hat{v},t) \in (s_1,s_2) \times (\hat{v}_1,\hat{v}_2) \times (t_1,t_2)\}, \tag{4.53}$$

where $\hat{f}(s,\hat{v},t) = f(s,\hat{v}e^{\lambda t},t)$ and $\hat{h}(s,\hat{v},t) = h(s,\hat{v}e^{\lambda t},t)$. The left-hand side reduces to a PDE which has derivatives in only two of the variables. We shall write $-\hat{\mathcal{L}}\hat{f} + \hat{h} = 0$ to denote (4.52).

(ii) For our purposes, $u(s,v,t)$ plays the role of f in $-\mathcal{L}f+h=0$ and the set (4.51) should sit inside the continuation region of the optimal stopping problem and h contains the integral term of the equation. Explicitly, h and $\hat{h}$ are given by

$$h(s,v,t) = \lambda \int_0^\infty u(se^{-\rho z}, v+z, t)\ \Pi(\mathrm{d}z),$$
$$\hat{h}(s,\hat{v},t) = \lambda \int_0^\infty u(se^{-\rho z}, \hat{v}e^{\lambda t}+z, t)\ \Pi(\mathrm{d}z).$$

Although $u(s,v,t)$ is only a viscosity solution of the equation $\mathcal{L}f = h$ inside the continuation region, it is well known that the viscosity solution property is preserved under a change of chart.

This is because the notion of viscosity solution is based on comparison of u against smooth test functions. If f is a subsolution, ϕ is a test function such that $u-\phi$ achieves a maximum at some point (s,v,t) with $f(s,v,t) = \phi(s,v,t)$, then $-\mathcal{L}\phi(s,v,t) + h(s,v,t) \le 0$. Clearly this holds if and only if the function $\hat{f}-\hat{\phi}$ achieves a maximum at some point $(s,\hat{v},t)$ with $u(s,\hat{v},t) = \phi(s,\hat{v},t)$, then -$\hat{\mathcal{L}}\hat{\phi}(s,\hat{v},t) + \hat{h}(s,\hat{v},t) \le 0$.

(iii) We chose an open ball B centred at $(s_0,\hat{v}_0,t_0)$ with radius R_0, which sits inside the cuboid (4.53). Let $B_\nu = B\cap\{\hat{v}=\nu\}$. Explicitly,

$$\left\{(s,t) : (s-s_0)^2 + (t-t_0)^2 \le \sqrt{R_0^2 - (\nu-\hat{v}_0)^2}\right\}$$

We set $R_\nu = \sqrt{R_0^2-(\nu-v_0)^2}$ to be the radius of B_ν. We can analyse (4.52) on B_ν by treating $\hat{v}$ as a parameter rather than a variable, with $\nu \in (\hat{v}_0 - R_0, \hat{v}_0+R_0)$. To emphasise this, we write

$$-\partial_t\hat{f}(s,t;\nu)+\frac{1}{2}s^2\nu e^{\lambda t}\partial_{ss}\hat{f}(s,t;\nu)+(r-C_\rho)s\partial_s\hat{f}(s,t;\nu)-(r+C_1)\hat{f}(s,t;\nu) = \hat{h}(s,t;\nu). \tag{4.54}$$

We use $\hat{L}_\nu\hat{f}(s,t;\nu) = \hat{h}(s,t;\nu)$ to denote (4.54). We stress the difference between $\hat{L}$ and $\hat{L}_\nu$ is that the former is seen as a (degenerate elliptic) operator acting on functions with 3 variables in a sphere, but the latter is a (uniformly) parabolic operator acting on functions with only 2 variables on a disc.

(iv) The function $\hat{h}(s,t;\nu)$ is $\frac{1}{2}$-Hölder continuous on B_ν. By Proposition 4.40, we have that

$$
\begin{aligned}
&|\hat{h}(s,t;\nu)-\hat{h}(s',t';\nu)| \\
&=\int_0^\infty |u(se^{-\rho z},\nu e^{\lambda t}+z,t)-u(s'e^{-\rho z},\nu e^{\lambda t'}+z,t')|\Pi(\mathrm{d}z) \\
&\leq C_0\Big(\int_0^\infty e^{-\rho z}|s-s'|+\nu|e^{\lambda t}-e^{\lambda t'}|+\nu^{\frac{1}{2}}|e^{\lambda t}-e^{\lambda t'}|^{\frac{1}{2}}\Pi(\mathrm{d}z) \\
&\quad+\int_0^\infty (1+\nu^{\frac{1}{2}}e^{\frac{\lambda t}{2}}+\nu e^{\lambda t}+z+z^{\frac{1}{2}})(|t-t'|^{\frac{1}{2}}+|t-t'|)\Pi(\mathrm{d}z)\Big) \\
&\leq C_0(|s-s'|+(1+\nu^{\frac{1}{2}}+\nu)(|t-t'|^{\frac{1}{2}}+|t-t'|)) \\
&\leq C_0(|s-s'|+|t-t'|^{\frac{1}{2}}),
\end{aligned}
$$

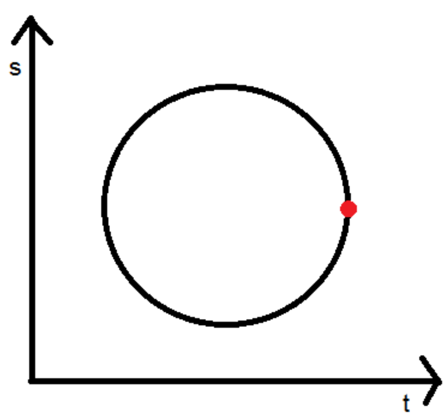
where C_0 is independent of ν, t, t', s, s' over the whole of B (since B is bounded).

(v) We consider the problems

$$\hat{L}_\nu \hat{f}(s,t;\nu)=\hat{h}(s,t;\nu) \qquad \text{for } (s,t)\in B_\nu \tag{4.55}$$

$$\hat{f}(s,t;\nu)=\hat{u}(s,\nu,t) \qquad \text{for } (s,t)\in \partial B_\nu \tag{4.56}$$

This is a parabolic problem. To guarantee the existence of a solution, we cannot impose a boundary condition at terminal time. In this case, it means, we cannot impose a condition at $(s_0, t_0+\sqrt{R_0^2-(\nu-\hat{v}_0)^2})$. The diagram below illustrates B_ν with the red dot marking $(s_0, t_0+\sqrt{R_0^2-(\nu-\hat{v}_0)^2})$.



So, if we just consider the problem

$$
\begin{aligned}
\hat{L}_\nu \hat{f}(s,t;\nu)&=\hat{h}(s,t;\nu) && \text{for } (s,t)\in B_\nu, \\
\hat{f}(s,t;\nu)&=\hat{u}(s,\nu,t) && \text{for } (s,t)\in \partial B_\nu \setminus (s_0, t_0+\sqrt{R_0^2-(\nu-\hat{v}_0)^2}),
\end{aligned}
$$

then this domain satisfies the outside sphere condition for Theorem 9 on page 69 of [28], hence it has a unique solution. This solution extends continuously to $(s_0, t_0 + \sqrt{R_0^2 - (\nu - \hat{v}_0)^2})$ and it is a unique solution to (4.55) and (4.56).

(vi) If we consider $\hat{f}(s, \hat{v}, t) = \hat{f}(s, t; \hat{v})$ for $(s, \hat{v}, t) \in B$, this defines a function in 3 variables. We want to show this is a viscosity solution of $-\hat{L}\hat{f} + \hat{h} = 0$. For $\hat{f}$ to be a viscosity solution, $\hat{f}$ needs to be continuous. We do this by showing locally $f(s, t; \nu)$ is uniformly continuous in ν.

We want to apply to Theorem 14 on page 80 of [28], which tells us that, if the coefficients of parabolic operators L_ϵ converges to coefficients of parabolic operators L_0 and h_ϵ converges h_0 on a domain D, provided the coefficients L_ϵ satisfies some uniformity conditions, then the solution of

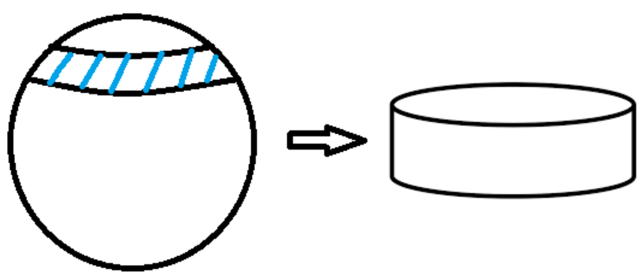
$$L_\epsilon \hat{f} = h_\epsilon$$

converges uniformly to the solution of $L_0 \hat{f} = \hat{h}_0$ in any closed subset of D.

We cannot apply this theorem directly to $\hat{L}_\nu$ because the domain B_ν changes according to the value of ν. However, we can perform a simple transformation of variable to transform a spherical segment to a cylinder with cross-sectional radius $R_{\hat{v}} = \sqrt{R_0^2 - (\hat{v} - \hat{v}_0)^2}$.

For a particular $\hat{v} \in (\hat{v}_0 - R_0, \hat{v}_0 + R_0)$, consider $\hat{f}(s, t; \nu)$ on B_ν for $\nu \in (\hat{v} - \epsilon, \hat{v} + \epsilon)$ such that $(\hat{v} - \epsilon, \hat{v} + \epsilon) \subset (v_0 - R_0, v_0 + R_0)$. For $\nu \in (\hat{v} - \epsilon, \hat{v} + \epsilon)$, consider the transformation

$$\tilde{s} = s_0 + \frac{R_{\hat{v}}}{R_\nu}(s - s_0), \quad \tilde{t} = t_0 + \frac{R_{\hat{v}}}{R_\nu}(t - t_0)$$



then, if we set

$$\tilde{f}(\tilde{s}, \tilde{t}; \nu) = \hat{f}\left(s_0 + \frac{R_\nu}{R_{\hat{v}}}(\tilde{s} - s_0), t_0 + \frac{R_\nu}{R_{\hat{v}}}(\tilde{t} - t_0); \nu\right),$$
$$\tilde{h}(\tilde{s}, \tilde{t}; \nu) = \hat{h}\left(s_0 + \frac{R_\nu}{R_{\hat{v}}}(\tilde{s} - s_0), t_0 + \frac{R_\nu}{R_{\hat{v}}}(\tilde{t} - t_0); \nu\right),$$

then $\tilde{f}$ and $\tilde{h}$ satisfies the equation

$$-\left(\frac{R_{\hat{v}}}{R_\nu}\right)\partial_2\tilde{f}(\tilde{s},\tilde{t};\nu)+\frac{1}{2}\left(s_0+\frac{R_\nu}{R_{\hat{v}}}(\tilde{s}-s_0)\right)^2\left(\frac{R_{\hat{v}}}{R_\nu}\right)^2\partial_{11}\tilde{f}(\tilde{s},\tilde{t};\nu)-(r+C_1)\tilde{f}(\tilde{s},\tilde{t};\nu)$$
$$+(r-C_\rho)\left(s_0+\frac{R_\nu}{R_{\hat{v}}}(\tilde{s}-s_0)\right)\left(\frac{R_{\hat{v}}}{R_\nu}\right)\partial_1\tilde{f}(\tilde{s},\tilde{t};\nu)=\tilde{h}(\tilde{s},\tilde{t};\nu)\quad \text{for } (\tilde{s},\tilde{t})\in B_{\hat{v}}$$

subject to the boundary condition

$$\tilde{f}(\tilde{s},\tilde{t};\nu)=\tilde{u}(\tilde{s},\tilde{t};\nu)\quad \text{for } (\tilde{s},\tilde{t})\in\partial B_{\hat{v}}$$

We can now apply Theorem 14 on page 80 of [28] to show that $\tilde{f}(\tilde{s},\tilde{t};\nu)$ is uniformly continuous in ν on any closed subset of $B_{\tilde{v}}$. This shows $\hat{f}(s,t;\nu)$ is continuous in ν, locally uniformly in (s,t).

We now prove Proposition 4.42.

Proof of Proposition 4.42. By [61], in the continuation region D^c, $u(s,v,t)$ is a viscosity solution of

$$-\partial_3 f(s,v,t)+\frac{1}{2}s^2v\partial_{11}f(s,v,t)-\lambda v\partial_2 f(s,v,t)+(r-C_\rho)s\partial_1 f(s,v,t)$$
$$-(r+C_1)f(s,v,t)+\lambda\int_0^\infty f(se^{-\rho z},v+z,t)\Pi(\mathrm{d}z)=0$$

where $C_1=\lambda\Pi(0,\infty)$. For values of (s,v,t) in the continuation region D^c, we can make a change of variables. By Remark 4.43 (i) - (ii), we have $-\hat{L}\hat{u}+\hat{h}=0$ in the viscosity sense in the transformed continuation region

$$\hat{C}=\{(s,\hat{v},t):\hat{u}(s,\hat{v},t)>g(s)\}.$$

We consider an open ball $B\subset\hat{C}$ centred at $(s_0,\hat{v}_0,t_0)$ with radius R_0. Let $B_\nu=B\cap\{\hat{v}=\nu\}$ be open discs centred at (s_0,ν,t_0). By treating $\hat{v}$ as a parameter we consider the family of parabolic PDE problems

$$\hat{L}_\nu\hat{f}(s,t;\nu)=\hat{h}(s,t;\nu)\qquad \text{for } (s,t)\in B_\nu, \tag{4.57}$$
$$\hat{f}(s,t;\nu)=\hat{u}(s,\nu,t)\qquad \text{for } (s,t)\in\partial B_\nu, \tag{4.58}$$

where $\hat{L}_\nu$ is given by the left hand side of (4.54). Since the coefficients of $\hat{L}_\nu$ are smooth and $\hat{h}$ is a Hölder continuous function in B_ν (Remark 4.43 (iv)), by the reasoning in Remark 4.43 (v), there is a families of unique classical solutions $\hat{f}(s,t;\nu)$ which solves (4.57) and (4.58).

We now define the function $\hat{f}(s,\hat{v},t) = \hat{f}(s,t;\hat{v})$. This satisfies the equation

$$-\hat{L}\hat{f}(s,\hat{v},t) + \hat{h}(s,\hat{v},t) = 0 \quad \text{for } (s,t) \in B$$

Moreover, $\hat{f}(s,\hat{v},t)$ agrees with $\hat{u}(s,\hat{v},t)$ on ∂B. By Remark 4.43 (vi), $\hat{u}(s,\hat{v},t)$ is continuous.

We now check that $\hat{f}(s,\hat{v},t)$ is a viscosity solution to $-\hat{\mathcal{L}}\hat{f} + \hat{h} = 0$ in B. Here, we use the elliptic definition of the superjets and subjets. Let $(p,A) \in J^{2,+}\hat{f}(s,\hat{v},t)$, where $J^{2,+}$ was defined on page 12. Since $\partial_{ss}\hat{f}(s,\hat{v},t)$, $\partial_s\hat{f}(s,\hat{v},t)$ and $\partial_t\hat{f}(s,\hat{v},t)$ exists, we must have

$$p_t = \partial_t\hat{f}(s,\hat{v},t),\ p_s = \partial_s\hat{f}(s,\hat{v},t),\ A_{ss} \geq \partial_{ss}\hat{f}(s,\hat{v},t).$$

The other components of the superjets do not feature in the equation so it does not matter what values they take. If we let p_t, p_s, A_{ss} take the place of $\hat{f}(s,\hat{v},t), \partial_s\hat{f}(s,\hat{v},t), \partial_{ss}\hat{f}(s,\hat{v},t)$ in (4.52), it is clear that $\hat{f}(s,\hat{v},t)$ is a viscosity subsolution. The supersolution property follows by a similar argument. By [19, Theorem 3.3], two viscosity solutions to the same degenerate elliptic problem on a bounded set Ω with the same boundary condition on $\partial\Omega$ must be equal. This means $\hat{f} = \hat{u}$ and the derivatives $\partial_s\hat{u}, \partial_{ss}\hat{u}, \partial_t\hat{u}$ exist. □

Remark 4.44. (i) By the argument in Remark 4.43 (2), $\hat{u}(s,\hat{v},t)$ for $(s,\hat{v},t) \in (0,\infty) \times (0,\infty) \times (0,\infty)$ is viscosity solution to the equation

$$\min((\partial_t - \hat{\mathcal{L}} + r)\hat{f}(s,\hat{v},t),\ \hat{f}(s,\hat{v},t) - g(s)) = 0,$$

where

$$\begin{aligned}\hat{\mathcal{L}}\hat{f}(s,\hat{v},t) = \frac{1}{2}s^2\hat{v}e^{\lambda t}\partial_{ss}\hat{f}(s,\hat{v},t) + (r - C_\rho)s\partial_s\hat{f}(s,\hat{v},t) \\ + \lambda\int_0^\infty \hat{u}(se^{-\rho z}, \hat{v} + ze^{-\lambda t}, t) - \hat{u}(s,\hat{v},t)\Pi(\mathrm{d}z)\end{aligned}$$

Proposition 4.42 suggests that the variational inequality satisfied by $\hat{u}$ may be a better choice than the variational inequality for $u(s,v,t)$. The reason is that the derivatives in the operator $(\partial_t - \mathcal{L} + r)\hat{f}(s,\hat{v},t)$ are known to exist in a classical sense everywhere except on the stopping boundary.

(ii) It is straightforward to check that $\hat{u}$ inherits many properties of u. This includes convexity in s, monotonicity in s, $\hat{v}$ and t. These properties imply that the transformed stopping surface $\hat{b}(\hat{v},t) = b(\hat{v}e^{\lambda t},t)$ is also decreasing in $\hat{v}$ and t. In Section 4.5.3, we will prove some properties about for this transformed stopping surface.

4.5.3 Continuity of stopping surface in finite horizon

In this section, we show that the transformed stopping surface $\hat{b}(\hat{v}, t)$ defined in Remark 4.44 (ii) is right-continuous in the t direction. For a fixed $\hat{v}$, when looking at $\hat{u}(s, \hat{v}, t)$ as a function of (s, t) only, then the arguments in Section 4.4.4 and Section 4.4.5 can be applied.

Remark 4.45. If we know $u(s, v, t)$ is smooth in the continuation region, then we can repeat the argument in Section 4.4.4 and Section 4.4.5 for finite horizon. However, as pointed out in Section 4.5.2, we do not know whether the derivatives $\partial_t u(s, v, t)$ and $\partial_v u(s, v, t)$ exist. The change of variable tells us that $\partial_t u(s, v, t) - \lambda v \partial_v u(s, v, t)$ is well-defined so we can use the variational inequality in the same way as we did in Section 4.4.4 and Section 4.4.5 subject to restrictions. Otherwise, the arguments are identical. We avoid repeating the argument in Section 4.4.4 and 4.4.5 and explain how adapt the arguments in Section 4.4.4 and Section 4.4.5 in the Chapter Appendix.

First we define β as

$$\beta(\hat{v}, t) = \lim_{t' \downarrow t} \hat{b}(\hat{v}, t')$$

Since $\hat{b}(\hat{v}, \cdot)$ is a decreasing function, $\beta(\hat{v}, t)$ is well defined with $\beta(\hat{v}, t) \leq \hat{b}(\hat{v}, t)$.

The following proposition is analogous to Proposition 4.35. $\hat{u}$ admits a smooth fit condition in s for a fixed value of $\hat{v}$.

Proposition 4.46. *$\hat{u}$ exhibits smooth pasting across the boundary in s for a fixed value of $\hat{v}$. That is to say for all $t \in (0, \infty)$, $\hat{v} \in (0, \infty)$,*

$$\lim_{D^c \ni (s', \hat{v}, t') \to (s', \hat{v}, t')} \partial_s \hat{u}(s, \hat{v}, t) = -1 \quad \text{for } s \in [\beta(\hat{v}, t), b(\hat{v}, t)].$$

Proposition 4.46 can then be used to obtain the following lemma, which is analogous to Lemma 4.37.

Lemma 4.47.

$$\lim_{D^c \ni (s', \hat{v}, t) \to (s', \hat{v}, t)} \frac{1}{2} s'^2 v' \partial_{11} u(s', \hat{v}, t') \geq K \left(r - \lambda \int_0^\infty e^{-\rho z} \Pi(dz) \right)$$

Proposition 4.46 and Lemma 4.47 can then be used to prove the following proposition, which is analogous to Proposition 4.38.

Proposition 4.48. *If r, λ, Π satisfy the condition*

$$C = r - \lambda \int_0^\infty e^{-\rho z} \Pi(dz),$$

then $\hat{b}(\hat{v}, t)$ is continuous in t.

The proofs for Proposition 4.46, Lemma 4.47 and Proposition 4.48 can be found on page 138.

4.6 Least Square Monte Carlo method for American put under the BNS model

The methods used in this section are based on Monte Carlo techniques. There are several different approaches when it comes to pricing American options by simulation. Some approaches try to approximate the stopping rule while others approximate the transition density by a discrete time Markov chain. For a detailed review of some of the Monte Carlo methods available for pricing American options, we refer to Chapter 8 of [31]. The method used in this section directly approximate the continuation value of a Bermudan option. This means solving a discrete time optimal stopping problem rather than a continuous time one. The solution of the discrete Bermudan option problem is obtained by implementation of the dynamic programming principle. The key is to approximate the conditional expectations by linear regressions.

The idea of approximating the continuation value of the Bermudan option by regression was studied by a number of authors. The most important ones include [13] by Carrière, [72] and [73] by Tsitsiklis and Van Roy, and [47] by Longstaff and Schwartz. Carrière's approach in [13] uses spline regression, where the explanatory variables are the state variables. In contrasts, the methods in [72], [73] and [47] only use least square regression, but a set of basis function of the state variables are used. In this section, we use Longstaff Schwartz method. Longstaff-Schwartz method is more popular among practitioners due to its better performance.

The estimated Bermudan option price is then useful for estimating the value of the American option with the same pay-off. The difference between the two is a source of error. Overall, there are three sources of error when approximating the value of an American option by a Bermudan option by Monte Carlo.

(i) The difference in price between American put and the corresponding Bermudan option.

(ii) The difference in distribution between the underlying continuous time process and the discretised process.

(iii) The Monte Carlo error made in estimating the Bermudan option by least square Monte Carlo method.

(iii) was discussed extensive in [17], where the authors obtained convergence rate which applies to our case. For (i), we show the price of Bermudan options converges to the price of the corresponding American options with a convergence of n^{-1} where n is the number of equally spaced interval. For (ii), there exist an exact sampling scheme when the jump measure is finite. We give this algorithm and prove its validity.

4.6.1 Error bound between American option and Bermudan option

In Proposition 4.6, we have already shown that the price of Bermudan options converge to the price of corresponding American option when we choose the exercise dates carefully. We now provide a bound on this difference for a general convex function.

Proposition 4.49. *Let $g(\cdot) : (0, \infty) \to \mathbb{R}$ be a convex function satisfying the following condition:*

(i) g' and g'' are well-defined except on a finite set $M = \{a_1, \dots, a_k\}$.

(ii) For $j = 1, \dots, k$, the limits

$$g'(a_j\pm) \overset{def}{=} \lim_{x \to a_j \pm} g'(x), \quad g''(a_j\pm) \overset{def}{=} \lim_{x \to a_j \pm} g''(x)$$

exist and are finite.

Recall the definition of U and u given by (4.8) and (4.7), then

$$u(s, v, T) - U(s, v, T, n) \le \frac{C_1(T, s, v)}{n},$$

Proof. For the first part of the proof, we assume that $g(0) \le 0$. Let τ be an arbitrary stopping time and define $\hat{\tau}$ by

$$\hat{\tau} = \inf \left\{ t \in \left\{ \frac{T}{n}, \dots, \frac{(n-1)T}{n}, T \right\} : t \ge \tau \right\}.$$

Then, we have that

$$U(s, v, T, n) \ge \mathbb{E} e^{-r\hat{\tau}} g(S_{\hat{\tau}}^{s,v}).$$

By the definition of $\hat{\tau}$, we have that $0 \le \tau - \hat{\tau} < \frac{T}{n}$, then

$$e^{-r\tau} - e^{-r\hat{\tau}} \le e^{-r\tau}(1 - e^{-r(\hat{\tau} - \tau)}) \le e^{-r\tau}(1 - e^{-\frac{rT}{n}}) \le \frac{rT}{n} \tag{4.59}$$

Let $L_-^S(t, a_i)$ denote the local time of $S^{s,v}$ from the left at a_i, defined by

$$L_-^S(t, a_i) = \lim_{\epsilon \downarrow 0} \int_0^t \mathbf{1}_{\{a_i - \epsilon \le S_q^{s,v} \le a_i\}} \mathrm{d}q.$$

By Itô's formula, we have that

$$
\begin{aligned}
&\mathbb{E}[e^{-r\tau}g(S^{s,v}_\tau) - e^{-r\hat{\tau}}g(S^{s,v}_{\hat{\tau}})] \\
&\quad\le \mathbb{E}\Big[-\int_\tau^{\hat{\tau}} \frac{1}{2}\underbrace{(S^{s,v}_t)^2V^v_t g''(S^{s,v}_t)}_{\ge 0} + \underbrace{rS^{s,v}_t g'(S^{s,v}_u) - rg(S^{s,v}_u)}_{\ge -rg(0)\ge 0 \text{ by convexity of } g}\mathrm{d}t\Big]+ \\
&\qquad \mathbb{E}\Big[-\int_{t=\tau}^{\hat{\tau}} \underbrace{g(S_te^{-\rho z}) - g(S^{s,v}_t) + S^{s,v}_t(1-e^{-\rho z})g'(S^{s,v}_t)}_{\ge 0 \text{ by convexity of } g}\Pi(\mathrm{d}z)\Big]+ \\
&\qquad \mathbb{E}\Big[-\frac{1}{2}\sum_{i=1}^k (e^{-r\hat{\tau}}L^S_-(\hat{\tau},a_i) - e^{-r\tau}L^S_-(\tau,a_i))\underbrace{(g'(a_i+) - g'(a_i-))}_{\ge 0}\Big] \\
&\quad\le \frac{1}{2}\mathbb{E}\Big[\sum_{i=1}^k \underbrace{(e^{-r\tau} - e^{-r\hat{\tau}})}_{\le \frac{rT}{n} \text{ by (4.59)}}\underbrace{L^S_-(\tau,a_i)}_{\le L^S_-(T,a_i)}(g'(a_i+) - g'(a_i-))\Big] \\
&\quad\le \frac{rT}{2n}\sum_{i=1}^k \mathbb{E}[L^S_-(T,a_i)](g'(a_i+) - g'(a_i-)),
\end{aligned}
$$

where the penultimate line follows from its preceding line because

$$
-(e^{-r\hat{\tau}}L^S_-(\hat{\tau},a_i) - e^{-r\tau}L^S_-(\tau,a_i)) = (e^{-r\tau} - e^{-r\hat{\tau}})L^S_-(\tau,a_i) + e^{-r\hat{\tau}}(\underbrace{L^S_-(\tau,a_i) - (L^S_-(\hat{\tau},a_i)}_{\le 0}).
$$

By choosing τ to be an optimal stopping time such that

$$
u(s,v,T) = \mathbb{E}e^{-r\tau}g(S^{s,v}_\tau), \tag{4.60}
$$

we have the following inequality

$$
\begin{aligned}
u(s,v,T) - U(s,v,T,n) &\le \mathbb{E}[e^{-r\tau}g(S^{s,v}_\tau) - e^{-r\hat{\tau}}g(S^{s,v}_{\hat{\tau}})] \\
&\le \frac{rT}{2n}\sum_{i=1}^k \mathbb{E}[L^S_-(T,a_i)](g'(a_i+) - g'(a_i-)).
\end{aligned}
$$

If $g(0)\ge 0$, consider the function $\tilde{g}(x) = g(x) - g(0)$. The previous argument holds for $\tilde{g}$ and any stopping time τ. Again, we choose τ to be stopping time such that (4.60) is satisfied. (Note τ is still the optimal stopping time for optimal stopping problem with pay-off g, not

$\tilde{g}$.) The error bound is now

$$\begin{aligned}
u(s,v,T) - U(s,v,T,n) \leq & \mathbb{E}e^{-r\tau} g(S^{s,v}_{\tau}) - e^{-r\hat{\tau}} g(S^{s,v}_{\hat{\tau}}) \\
= & \mathbb{E}e^{-r\tau} \tilde{g}(S^{s,v}_{\tau}) - e^{-r\hat{\tau}} \tilde{g}(S^{s,v}_{\hat{\tau}}) + g(0)\mathbb{E}[e^{-r\tau} - e^{-r\hat{\tau}}] \\
\leq & \frac{rT}{2n} \sum_{i=1}^{k} \mathbb{E}[L^S_{-}(T,a_i)](\tilde{g}'(a_i+) - \tilde{g}'(a_i-)) + \frac{rTg(0)}{n} \\
= & \frac{rT}{2n} \sum_{i=1}^{k} \mathbb{E}[L^S_{-}(T,a_i)](g'(a_i+) - g'(a_i-)) + \frac{rTg(0)}{n}
\end{aligned}$$

where the last equality holds because $g'(a_i\pm) = \tilde{g}'(a_i\pm)$ for $a_i \in M$. Hence,

$$u(s,v,T) - U(s,v,T,n) \leq \frac{C_1(T,s,v)}{n},$$

where

$$C_1(T,s,v) = \frac{rT}{2} \sum_{i=1}^{k} \mathbb{E}[L^S_{-}(T,a_i)](g'(a_i+) - g'(a_i-)) + rT\max(0, g(0)) \tag{4.61}$$

□

In the case of the American put problem, $k = 1$ and $a_1 = K$.

4.6.2 Exact simulation scheme for the BNS model

When simulating an SDE with jumps, discretisation errors are often introduced. The marginal distributions of the simulated process often differ to the marginal distributions of the underlying process at the same time points. In the case of the BNS model, there exists an exact sampling scheme. The distribution of the simulated process under this scheme coincides with the distribution of the (S, V), provided the jump measure is finite.

When the jump measure is finite, we can normalise Π. Define $\hat{\Pi}$ by

$$\hat{\Pi}(\mathrm{d}z) = \frac{\Pi(\mathrm{d}z)}{\Pi(0,\infty)} \tag{4.62}$$

Let $(\hat{S}^{s,v}, \hat{V}^v)$ denote simulated process with initial values $\hat{S}_0 = s$ and $\hat{V}_0 = v$. A sampling scheme for generating $(\hat{S}^{s,v}_t, \hat{V}^v_t)_{t\in\{\frac{T}{n}, \frac{2T}{n}, \ldots, T\}}$, in the case where the jump measure is finite, is given as follows.

(i) Set $\hat{S}_0 = s$ and $\hat{V}_0 = v$, start a counter $i = 0$.

(ii) Sample $\hat{N}$ from Poisson distribution with mean $\frac{\lambda\Pi(0,\infty)T}{n}$.

(iii) If $\hat{N} > 0$, then generate an independent random sample $J_1, \ldots, J_{\hat{N}}$ of size $\hat{N}$ from the distribution $\hat{\Pi}$ given by (4.62). We then generate an independent random sample $T_1, T_2, \ldots, T_{\hat{N}}$ of size $\hat{N}$ from uniform distribution on the interval $(0, \frac{T}{n})$. Set $\hat{V}_{\frac{(i+1)T}{n}}$ as:

$$\hat{V}_{\frac{(i+1)T}{n}} = \hat{V}_{\frac{iT}{n}} e^{-\frac{\lambda T}{n}} + \sum_{i=1}^{\hat{N}} J_i e^{-\lambda(\frac{T}{n} - T_i)},$$

where the summation term is taken to be 0 if $\hat{N} = 0$.

(iv) Define U to be

$$\hat{U} = \frac{\hat{V}_{\frac{iT}{n}}}{\lambda}(1 - e^{-\frac{\lambda T}{n}}) + \sum_{i=1}^{\hat{N}} \frac{J_i}{\lambda}(1 - e^{-\lambda(\frac{T}{n} - T_i)}),$$

(v) Generate normal random variable $N_{\hat{U}}$ such that it has mean $-\frac{1}{2}\hat{U}$ and variance $\hat{U}$.

(vi) Set $\hat{S}_{\frac{(i+1)T}{n}}$ as:

$$\hat{S}_{\frac{(i+1)T}{n}} = \hat{S}_{\frac{iT}{n}} \exp\left(N_U + \frac{(r + \lambda\Phi(\rho))T}{n} - \rho \sum_{i=1}^{\hat{N}} J_i\right)$$

(vii) Add 1 to the counter i. If $i < n$, then we repeat steps (ii) to (vii), else we are done.

We now show the distribution of the simulated process coincide with the distribution of the underlying process.

Proposition 4.50. *The process*

$$\{(\hat{S}^{s,v}_{\frac{iT}{n}}, \hat{V}^{v}_{\frac{iT}{n}}) : 0 \leq i \leq n\},$$

simulated using steps (i) to (vii) above has the same distribution as

$$\{(S^{s,v}_{\frac{iT}{n}}, V^{v}_{\frac{iT}{n}}) : 0 \leq i \leq n\}.$$

Proof. We first show that $\{\hat{V}_{\frac{iT}{n}} : i = 0, \ldots, n\}$ has the same distribution as $\{V_{\frac{iT}{n}} : i = 0, \ldots, n\}$. We do this by induction on j. We show for $j = 0, 1, \ldots,$ $\{\hat{V}_{\frac{iT}{n}} : i = 0, \ldots, j\}$ and $\{V_{\frac{iT}{n}} : i = 0, \ldots, j\}$ have the same distribution.

For $i = 0$, the proposition is trivial since $V^v_0 = \hat{V}^v_0 = v$. Assume that $\{\hat{V}_{\frac{iT}{n}} : i = 0, \ldots, j\}$ and $\{V_{\frac{iT}{n}} : i = 0, \ldots, j\}$ have the same distribution for some j. We set $t = \frac{(j+1)T}{n}$ and $t' = \frac{jT}{n}$ in the summation representation given by (4.4) on page 85. This allows us to arrive at the equation:

$$V^v_{\frac{(j+1)T}{n}} = V^v_{\frac{jT}{n}} e^{-\frac{\lambda T}{n}} + \sum_{\frac{jT}{n}<u\le\frac{(j+1)T}{n}} \Delta Z_u e^{-\lambda(\frac{(j+1)T}{n}-u)},$$

Conditional on $\{V_{\frac{iT}{n}} : i=0,\ldots,j\}$, the distribution of $V^v_{\frac{(j+1)T}{n}}$ is $V^v_{\frac{jT}{n}} e^{-\frac{\lambda T}{n}}$ plus the sum

$$\sum_{\frac{jT}{n}<u\le\frac{(j+1)T}{n}} \Delta Z_u e^{-\lambda(\frac{(j+1)T}{n}-u)}, \tag{4.63}$$

which is independent of $\{V_{\frac{iT}{n}} : i=0,\ldots,j\}$.

Since $\Pi(0,\infty)<\infty$, the number of jumps in the period $(\frac{jT}{n},\frac{(j+1)T}{n}]$ is finite almost surely. The number of jumps is the same as the number of terms in (4.63). If we denote the number of terms in (4.63) by N and the jump times by $t_1,\ldots,t_N$, then

$$V^v_{\frac{(j+1)T}{n}} = V^v_{\frac{jT}{n}} e^{-\frac{\lambda T}{n}} + \sum_{i=1}^{N} \Delta Z_{t_i} e^{-\lambda(\frac{(j+1)T}{n}-t_i)},$$

N is known to have a Poisson distribution with mean $\frac{\lambda\Pi(0,\infty)T}{n}$. Conditional on N, the jump times $\{t_1,\ldots,t_N\}$ are independently uniformly distributed over the time interval $(\frac{jT}{n},\frac{(j+1)T}{n}]$; the jump sizes, $\{\Delta Z_{t_i} : i=1,\ldots,N\}$, which are independent of $\{t_1,\ldots,t_N\}$, are independently identically distributed according to $\hat{\Pi}(\mathrm{d}z)$.

In step (ii) and (iii) of the sampling scheme, $\hat{N}$ is generated from the same distribution as N. Conditional on $\hat{N}$, $\{T_1+\frac{jT}{n},\ldots,T_{\hat{N}}+\frac{jT}{n},J_1,\ldots,J_{\hat{N}}\}$ have the same distribution as $\{t_1,\ldots,t_N,\Delta Z_{t_i},\ldots,\Delta Z_{t_N}\}$ conditional on $\hat{N}$. Hence, $\hat{V}_{\frac{(j+1)T}{n}}$ conditional on $\{\hat{V}_{\frac{iT}{n}} : i=0,\ldots,j\}$ has the same distribution as $V_{\frac{(j+1)T}{n}}$ conditional on $\{V_{\frac{iT}{n}} : i=0,\ldots,j\}$. By induction hypothesis $\{\hat{V}_{\frac{iT}{n}} : i=0,\ldots,j\}$ and $\{V_{\frac{iT}{n}} : i=0,\ldots,j\}$ have the same distribution , hence $\{\hat{V}_{\frac{iT}{n}} : i=0,\ldots,j+1\}$ and $\{V_{\frac{iT}{n}} : i=0,\ldots,j+1\}$ have the same distribution.

We now show that $\{(\hat{S}_{\frac{iT}{n}},\hat{V}_{\frac{iT}{n}}) : i=0,\ldots,n\}$ has the same distribution as $\{(S_{\frac{iT}{n}},V_{\frac{iT}{n}}) : i=0,\ldots,n\}$ by induction on j. Again, $S^{s,v}_0=\hat{S}^{s,v}_0=s$ so the case $j=0$ holds trivially. We assume $\{(\hat{S}_{\frac{iT}{n}},\hat{V}_{\frac{iT}{n}}) : i=0,\ldots,j\}$ has the same distribution as $\{(S_{\frac{iT}{n}},V_{\frac{iT}{n}}) : i=0,\ldots,j\}$ for some j. For $\frac{jT}{n}\le t\le\frac{(j+1)T}{n}$, by rearranging (4.4), we arrive at the expression

$$V^v_t = V^v_{\frac{jT}{n}} e^{-\lambda(t-\frac{jT}{n})} + \sum_{i=1}^{N} \Delta Z_{t_i} e^{-\lambda(t-t_i)} \mathbf{1}_{\{t\ge t_i\}}. \tag{4.64}$$

Let U be the integral of V^v_t on the interval $(\frac{jT}{n},\frac{(j+1)T}{n})$, then from (4.64) we obtain the following expression.

$$U \stackrel{\text{def}}{=} \int_{\frac{jT}{n}}^{\frac{(j+1)T}{n}} V_u \mathrm{d}u = \frac{V_{\frac{jT}{n}}}{\lambda}(1 - e^{-\frac{\lambda T}{n}}) + \sum_{i=1}^{N} \frac{\Delta Z_{t_i}}{\lambda}(1 - e^{-\lambda(\frac{(j+1)T}{n} - t_i)}). \tag{4.65}$$

Conditional on

$$\{N, t_1, \ldots, t_N, \Delta Z_{t_1}, \ldots, \Delta Z_{t_N}, (S_0^{s,v}, V_0^{s,v}), \ldots, (S_{\frac{jT}{n}}^{s,v}, V_{\frac{jT}{n}}^{v})\}, \tag{4.66}$$

$X_{\frac{(j+1)T}{n}}$ is normally distributed with mean

$$X_{\frac{jT}{n}} - \frac{1}{2}U + \frac{(r + \lambda\Phi(\rho)T}{n} - \rho\sum_{i=1}^{N} \Delta Z_{t_i} \tag{4.67}$$

and variance U. From (4.67), we deduce that the distribution of $S_{\frac{(j+1)T}{n}}$ conditional on the set of random variables labeled as (4.66) is lognormal. $\hat{U}$ generated in steps (iv) of the simulation scheme has the same distribution as U. Step (v) and (vi) of the sampling scheme calculate $\hat{S}_{\frac{(j+1)T}{n}}$ from its conditional distribution on

$$\{\hat{N}, T_1, \ldots, T_{\hat{N}}, J_1, \ldots, J_{\hat{N}}, (\hat{S}_0^{s,v}, \hat{V}_0^{s,v}), \ldots, (\hat{S}_{\frac{jT}{n}}^{s,v}, \hat{V}_{\frac{jT}{n}}^{v})\},$$

which is the same distribution as the conditional distribution of $S_{\frac{(j+1)T}{n}}$ on the list of random variables given by (4.66). Hence, $\{(\hat{S}_{\frac{iT}{n}}, \hat{V}_{\frac{iT}{n}}) : i = 0, \ldots, j+1\}$ has the same distribution as $\{(S_{\frac{iT}{n}}, V_{\frac{iT}{n}}) : i = 0, \ldots, j+1\}$ and we are done. □

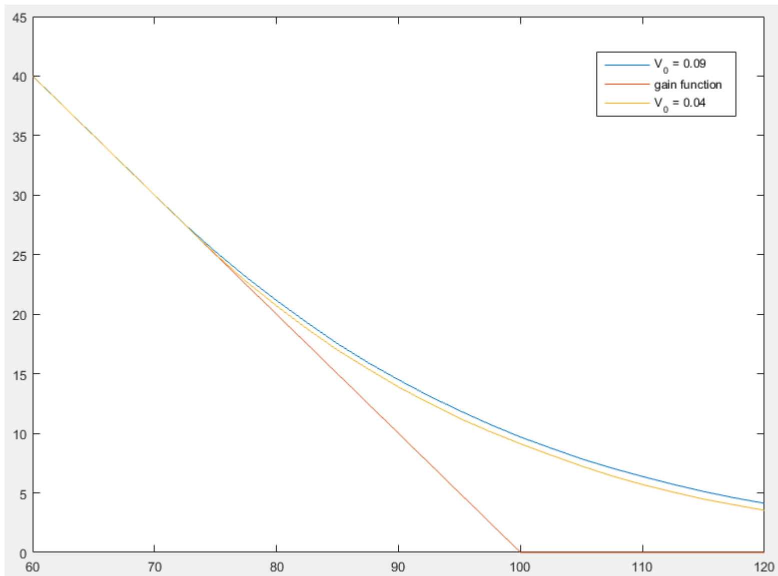
Example 4.51. We illustrate the Least Square Monte Carlo method described in this section. The parameters we have chosen are $r = 0.05, T = 1, \lambda = 6, \rho = 1, K = 100$. The jump density Π is chosen to be $80000\exp(-400z)$, so the Lévy subordinator is a compound Poisson process with exponentially distributed jumps. To produce the estimates below, we used sample size of 20000 sample path with 50 time steps.

S_0	V_0	$U(s, v, 1, 50)$
80	0.09	21.16
85	0.09	17.53
90	0.09	14.50
95	0.09	11.91
100	0.09	9.69
105	0.09	7.86
110	0.09	6.38

S_0	V_0	$U(s, v, 1, 50)$
80	0.04	20.70
85	0.04	16.99
90	0.04	13.89
95	0.04	11.26
100	0.04	7.26
105	0.04	5.70
110	0.04	4.48

The basis functions we used are S, V, SV, S^2 and V^2 as well as constants. We tried adding third order polynomial terms, but they did not change the estimates of the value function by a significant amount and produced warning messages in Matlab. The warning messages comes from matrix inversion procedure when performing the regressions. This is because the dependence on higher order terms are relatively weak and we should not include for stability purposes.

From this result, we are able to produce the following diagram.



For $v = 0.09$, $b(0.09, 1)$ appears be around 72. For $v = 0.04$, the stopping boundary appears to be around 75. The estimated value function appears to be convex in the stock price and increasing in squared volatility. This is consistent with our theoretical result.

4.7 Chapter Appendix

This section contains all of the proofs, which were omitted in Section 4.1 - 4.6. Here are the reasons why the proofs are found in this section rather than beneath the statement of the corresponding propositions.

(i) The proof is a relatively standard.

(ii) A very similar proof has already been presented in the chapter or another referenced source.

(iii) The result is a technical auxiliary lemma to a proposition we wish to prove.

Proof of Proposition 4.6 The proof of this lemma is relatively standard and is similar to the proof of Lemma 2.21.

Proof. Observe that for $n > m$, we have

$$U(s,v,T,2^m) \leq U(s,v,T,2^n) \leq u(s,v,T).$$

This means the limit of the left-hand side of (4.9) exists as it is the limit of a monotone sequence. Moreover, $\lim_{m\to\infty} U(s,v,T,2^m) \leq u(s,v,T)$. Since the time horizon is finite, there is an optimal stopping time τ such that

$$u(s,v,T) = \mathbb{E}e^{-r\tau}g(S_\tau^{s,v}).$$

Define the set $T_m = \{2^{-m}T, 2\cdot 2^{-m}T, ..., (2^m-1)\cdot 2^{-m}T, T\}$ and $\tau^m = \inf\{t \in T_m : t \geq \tau\}$. Note that $\tau^m \to \tau$ almost surely as $m \to \infty$, so we have

$$\begin{aligned} u(s,v,T) - U(s,v,T,2^m) \leq& \mathbb{E}e^{-r\tau}g(S_\tau^{s,v}) - \mathbb{E}e^{-r\tau^m}g(S_{\tau^m}^{s,v}) \\ \leq& \mathbb{E}|e^{-r\tau}g(S_\tau^{s,v}) - e^{-r\tau^m}g(S_{\tau^m}^{s,v})| \to 0, \end{aligned}$$

where the convergence holds by dominated convergence using the Assumption 4.4. Since ϵ is arbitrary, this proves that $\lim_{m\to\infty} U(s,v,T,2^m) \geq u(s,v,T)$ and (4.9) follows. □

Proof of Lemma 4.9

Proof. Let K be a constant such that $|g| < K$. Suppose τ_ϵ is an ϵ-optimal stopping time for $u(s,v,\infty)$, then consider

$$\tau_\epsilon^T = \tau_\epsilon \mathbf{1}_{\{\tau_\epsilon < T\}} + T\mathbf{1}_{\{\tau_\epsilon \geq T\}}$$

Since $\tau_\epsilon^T < T$, τ_ϵ^T is feasible stopping time for the finite horizon problem with time horizon T. If follows that

$$\mathbb{E}e^{-r\tau_\epsilon^T}g(S_{\tau_\epsilon^T}) \leq u(s,v,T) \tag{4.68}$$

Using the definition of ϵ-optimal time, we have

$$\begin{aligned}
u(s,v,\infty) \leq & \mathbb{E}e^{-r\tau_\epsilon} g(S^{s,v}_{\tau_\epsilon}) + \epsilon \\
= & \mathbb{E}e^{-r\tau_\epsilon} g(S^{s,v}_{\tau_\epsilon})\mathbf{1}_{\{\tau_\epsilon < T\}} + \mathbb{E}e^{-r\tau_\epsilon} g(S^{s,v}_{\tau_\epsilon})\mathbf{1}_{\{\tau_\epsilon \geq T\}} + \epsilon \\
\leq & \underbrace{\mathbb{E}e^{-r\tau_\epsilon} g(S^{s,v}_{\tau_\epsilon})\mathbf{1}_{\{\tau_\epsilon < T\}} + \mathbb{E}e^{-rT} g(S_T)\mathbf{1}_{\{\tau_\epsilon \geq T\}}}_{=\mathbb{E}e^{-r\tau^T_\epsilon} g(S^{s,v}_{\tau^T_\epsilon})} - \mathbb{E}e^{-rT} g(S^{s,v}_T)\mathbf{1}_{\{\tau_\epsilon \geq T\}} \\
& + \mathbb{E}e^{-r\tau_\epsilon} g(S^{s,v}_{\tau_\epsilon})\mathbf{1}_{\{\tau_\epsilon \geq T\}} + \epsilon \\
\leq & \mathbb{E}e^{-r\tau^T_\epsilon} g(S^{s,v}_{\tau^T_\epsilon}) + 2e^{-rT}K + \epsilon \\
\leq & u(s,v,T) + 2e^{-rT}K + \epsilon,
\end{aligned}$$

where the final line follows from the penultimate line by (4.68) and the penultimate line follows its preceding line by the bounds

$$|\mathbb{E}e^{-r\tau_\epsilon} g(S_{\tau_\epsilon})\mathbf{1}_{\{\tau_\epsilon \geq T\}}| \leq \mathbb{E}e^{-r\tau_\epsilon} K\mathbf{1}_{\{\tau_\epsilon \geq T\}} \leq Ke^{-rT}$$

and

$$|\mathbb{E}e^{-rT} g(S_{\tau_T})\mathbf{1}_{\{\tau_\epsilon \geq T\}}| \leq \mathbb{E}e^{-rT} K\mathbf{1}_{\{\tau_\epsilon \geq T\}} \leq Ke^{-rT}.$$

In addition, it is clear that $u(s,v,T)$ is increasing in T. This is because we are taking supremum over increasingly larger set of stopping times as T increases. This allows us to conclude that

$$u(s,v,T) \leq u(s,v,\infty) \leq u(s,v,T) + 2Ke^{-rT} + \epsilon$$

Since ϵ is arbitrary, it follows that

$$\lim_{T\to\infty} u(s,v,T) = u(s,v,\infty).$$

□

Derivation for generator of (S,V). Let us assume that $f \in C^{2,1}(\mathcal{O})$ with bounded derivatives. We wish to calculate

$$\lim_{t'\downarrow t} \frac{1}{t'-t}\mathbb{E}[f(S_{t'},V_{t'}) - f(S_t,V_t)|\mathcal{F}_t]$$

By Itô's formula, for $t' > t$, we have that

$$
\begin{aligned}
&f(S_{t'}, V_{t'}) \\
&\quad = f(S_t, V_t) \\
&\quad + \int_t^{t'} \partial_1 f(S_{u-}, V_{u-})\ \mathrm{d}S_u + \frac{1}{2}\int_t^{t'} \partial_{11} f(S_{u-}, V_{u-}) S_{u-}^2 V_{u-}\ \mathrm{d}u + \int_t^{t'} \partial_2 f(S_{u-}, V_{u-})\ \mathrm{d}V_u \\
&\quad + \sum_{t<u\le t'} [f(S_u, V_u) - f(S_{u-}, V_{u-}) - \Delta V_u \partial_2 f(S_{u-}, V_{u-}) - \Delta S_u \partial_1 f(S_{u-}, V_{u-})] \\
&\quad = f(S_t, V_t) + \int_t^{t'} r S_{u-} \partial_1 f(S_{u-}, V_{u-})\ \mathrm{d}u + \int_t^{t'} e^{ru} \partial_1 f(S_{u-}, V_{u-})\ \mathrm{d}(e^{-ru} S_u) \\
&\quad + \frac{1}{2}\int_t^{t'} \partial_{11} f(S_u, V_u) S_u^2 V_u\ \mathrm{d}u - \int_t^{t'} \partial_2 f(S_{u-}, V_{u-}) \lambda V_u\ \mathrm{d}u + \int_t^{t'} \partial_2 f(S_{u-}, V_{u-})\ \mathrm{d}Z_{\lambda u} \\
&\quad + \sum_{t<u\le t'} [f(S_u, V_u) - f(S_{u-}, V_{u-}) - \Delta V_u \partial_2 f(S_{u-}, V_{u-}) - \Delta S_u \partial_1 f(S_{u-}, V_{u-})],
\end{aligned}
$$

where we used (4.2) and

$$\mathrm{d}S_u = e^{ru}\ \mathrm{d}(e^{-ru} S_u) + r S_{u-}\ \mathrm{d}u.$$

Notice that $\Delta V_u = \mathrm{d}Z_{\lambda u}$, so

$$\int_t^{t'} \partial_2 f(S_{u-}, V_{u-})\ \mathrm{d}Z_{\lambda u} = \sum_{t<u\le t'} \Delta V_u \partial_2 f(S_{u-}, V_{u-}).$$

Moreover, we can write the sum

$$\sum_{t<u\le t'} [f(S_u, V_u) - f(S_{u-}, V_{u-}) - \Delta S_u \partial_1 f(S_{u-}, V_{u-})]$$

as an integral against the Lévy measure Π plus an integral against the compensated jump measure $\tilde{N}$. Therefore, $f(S_t, V_t)$ can be written as

$$
\begin{aligned}
&f(S_t, V_t) = f(S_{t'}, V_{t'}) \\
&+ \int_{t'}^{t} r S_u \partial_1 f(S_u, V_u)\ \mathrm{d}u - \int_{t'}^{t} \lambda \partial_2 f(S_{u-}, V_{u-}) V_u\ \mathrm{d}u + \frac{1}{2}\int_{t'}^{t} \partial_{11} f(S_{u-}, V_{u-}) S_{u-}^2 V_{u-}\ \mathrm{d}u \\
&+ \lambda \int_{u=t'}^{t} \int_{z=0}^{\infty} f(S_{u-} e^{-\rho z}, V_{u-} + z) - f(S_{u-}, V_{u-}) + S_{u-}(1 - e^{-\rho z}) \partial_1 f(S_{u-}, V_{u-}) \Pi(\mathrm{d}z)\mathrm{d}u \\
&+ \int_{u=t'}^{t} \int_{z=0}^{\infty} f(S_{u-} e^{-\rho z}, V_{u-} + z) - f(S_{u-}, V_{u-}) + S_{u-}(1 - e^{-\rho z}) \partial_1 f(S_{u-}, V_{u-}) \tilde{N}(\mathrm{d}z, \lambda \mathrm{d}u) \\
&+ \int_{t'}^{t} e^{ru} \partial_1 f(S_u, V_u)\ \mathrm{d}(e^{-ru} S_u).
\end{aligned}
$$

Now note the last two terms of the above expansion are martingales because f has bounded derivatives, so their conditional expectation with respect to $\mathcal{F}_t$ is 0.

It follows that

$$
\begin{aligned}
&\lim_{t'\downarrow t}\frac{1}{t'-t}\mathbb{E}[f(S_{t'},V_{t'})-f(S_t,V_t)|\mathcal{F}_t]\\
&=\lim_{t'\downarrow t}\frac{1}{t'-t}\mathbb{E}\bigg[\int_t^{t'}\frac{1}{2}S_u^2V_u\partial_{11}f(S_u,V_u)+rS_u\partial_1 f(S_u,V_u)-\lambda V_u\partial_2 f(S_u,V_u)\mathrm{d}u+\\
&\lambda\int_{u=t'}^{t}\int_{z=0}^{\infty}f(S_{u-}e^{-\rho z},V_{u-}+z)-f(S_{u-},V_{u-})+S_{u-}(1-e^{-\rho z})\partial_1 f(S_{u-},V_{u-})\Pi(\mathrm{d}z)\mathrm{d}u\bigg|\mathcal{F}_t\bigg]\\
&=\mathbb{E}\bigg[\lim_{t'\downarrow t}\frac{1}{t'-t}\bigg[\int_t^{t'}\frac{1}{2}S_u^2V_u\partial_{11}f(S_u,V_u)+rS_u\partial_1 f(S_u,V_u)-\lambda V_u\partial_2 f(S_u,V_u)\mathrm{d}u+\\
&\lambda\int_{u=t'}^{t}\int_{z=0}^{\infty}f(S_{u-}e^{-\rho z},V_{u-}+z)-f(S_{u-},V_{u-})+S_{u-}(1-e^{-\rho z})\partial_1 f(S_{u-},V_{u-})\Pi(\mathrm{d}z)\mathrm{d}u\bigg]\bigg|\mathcal{F}_t\bigg]\\
&=\frac{1}{2}S_t^2V_t\partial_{11}f(S_t,V_t)+rS_t\partial_1 f(S_t,V_t)-\lambda V_t\partial_2 f(S_t,V_t)\\
&\qquad+\lambda\int_0^{\infty}f(S_te^{-\rho z},V_t+z)-f(S_t,V_t)+S_t(1-e^{-\rho z})\partial_1 f(S_t,V_t)\Pi(\mathrm{d}z),
\end{aligned}
$$

where the limit exchange is justified by the dominated convergence theorem. This limit is time-homogeneous. By setting $t=0$, for $(s,v)\in\mathcal{O}$, we get the following infinitesimal generator for a strong Markov family:

$$
\begin{aligned}
Lf(s,v)=\frac{1}{2}s^2v\partial_{11}f(s,v)&+rs\partial_1 f(s,v)-\lambda v\partial_2 f(s,v)\\
&+\lambda\int_0^{\infty}f(se^{-\rho z},v+z)-f(s,v)+s(1-e^{-\rho z})\partial_1 f(s,v)\Pi(\mathrm{d}z).
\end{aligned}
$$

Proof of Corollary 4.21

(i) The left and right derivative must exist because $u(\cdot,v,T)$ is convex.

(ii) The upper bounds

$$\partial_1^- u(s,v,T)\le 0 \quad \text{for } s>0 \text{ and } \partial_1^+ u(s,v,T)\le 0 \quad \text{for } s\ge 0, \tag{4.69}$$

must hold because u is decreasing.

(iii) Since $u(b(v,T),v,T)=K-b(v,T)$, it must be the case that

$$\partial_1^+ u(b(v,T),v,T)\ge -1$$

else there would exist an ϵ, for $s\in(b_1,b_1+\epsilon)$ such that $u(s,v,T)<K-s$, which is

a contradiction. It follows by convexity of u that

$$\partial_1^- u(s,v,T) \geq -1 \quad \text{ for } s > b(v,T) \text{ and } \partial_1^+ u(s,v,T) \geq -1 \quad \text{ for } s \geq b(v,T).$$

(iv) For $s < b(v,T)$, $u(s,v,T) = K - s$, so

$$\partial_1^- u(s,v,T) = -1 \quad \text{ for } 0 < s \leq b(v,T) \text{ and } \partial_1^+ u(s,v,T) = -1 \quad \text{ for } 0 \leq s < b(v,T).$$

Proof of Proposition 4.24

This proof is very similar to [61, Thereom 4.3].

Proof. In this proof, C is a generic constant, which may change from line to line. Recall (4.6) and from this, we deduce that

$$\int_0^T V^{v+\epsilon} \mathrm{d}t = \int_0^T V^v \mathrm{d}t + \frac{\epsilon}{\lambda}(1 - e^{-\lambda T})$$

Moreover,

$$X_T^{x',v'} - X_T^{x,v} = (x - x') - \frac{1}{2\lambda}(v - v')(1 - e^{-\lambda T}) + \int_0^T \left(\sqrt{V_\tau^{v'}} - \sqrt{V_\tau^{v}}\right) \mathrm{d}B_t \tag{4.70}$$

By Lipschitz continuity of g, we have that

$$\begin{aligned}
|\tilde{u}(x', v', T) - \tilde{u}(x, v, T)| &= |\sup_{0 \leq \tau \leq T} \mathbb{E} e^{-r\tau} g(\exp(X_\tau^{x',v'})) - \sup_{0 \leq \tau \leq T} \mathbb{E} e^{-r\tau} g(\exp(X_\tau^{x,v})| \\
&\leq \sup_{0 \leq \tau \leq T} \mathbb{E}|g(\exp(X_\tau^{x',v'})) - g(\exp(X_\tau^{x,v})| \\
&\leq C \sup_{0 \leq \tau \leq T} \mathbb{E}|X_\tau^{x',v'} - X_\tau^{x,v}|,
\end{aligned}$$

where C at the moment is the Lipschitz constant for $g(\exp(\cdot))$. By (4.70), we arrive at

$$|\tilde{u}(x', v', T) - \tilde{u}(x, v, T)| \leq C\left(|x' - x| + |v' - v| + \sup_{0 \leq \tau \leq T} \mathbb{E}\left| \int_0^\tau \left(\sqrt{V_t^{v'}} - \sqrt{V_t^{v}}\right) \mathrm{d}B_t \right|\right) \tag{4.71}$$

By Cauchy-Schwarz and Doob's inequality, we have that

$$\begin{aligned}\sup_{0\le\tau\le T}\mathbb{E}\left|\int_0^\tau\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right| &\le \mathbb{E}\sup_{0\le\tau\le T}\left|\int_0^\tau\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right|\\ &\le \left(\mathbb{E}\sup_{0\le\tau\le T}\left|\int_0^\tau\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right|^2\right)^{1/2}\\ &\le 2\left(\mathbb{E}\left|\int_0^T\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right|^2\right)^{1/2}\end{aligned}$$

We now use Itô isometry on the last integral. We obtain

$$\begin{aligned}&\mathbb{E}\left|\int_0^T\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right|^2=\mathbb{E}\left|\int_0^T\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)^2\mathrm{d}t\ \right|\\ \le&\mathbb{E}\int_0^T|V_t^{v'}-V_t^{v}|\mathrm{d}t=\int_0^T|v-v'|e^{-\lambda t}\mathrm{d}t\le\frac{1}{\lambda}|v-v'|,\end{aligned}$$

where the inequality follows because $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$ for $a,b\ge 0$. Here we apply the inequality with $a=\min(V_t^{v'},V_t^{v})$ and $b=\max(V_t^{v'},V_t^{v})-\min(V_t^{v'},V_t^{v})$. This allows us to conclude that

$$\sup_{0\le\tau\le T}\mathbb{E}\left|\int_0^\tau\left(\sqrt{V_t^{v'}}-\sqrt{V_t^{v}}\right)\mathrm{d}B_t\right|\le\frac{2}{\sqrt{\lambda}}|v-v'|^{\frac{1}{2}} \tag{4.72}$$

The required result follows if we substitute (4.72) into (4.71). □

Proof of Proposition 4.28

This proof is very similar to [61, Thereom 4.3].

Proof. By Proposition 4.25, u is continuous. We begin by showing u is a subsolution of (4.26). Define the Markov time

$$\tau_D \stackrel{\text{def}}{=} \inf\{t:(S_t,V_t)\in D\}.$$

The value function is obtained at this Markov time since the gain function is bounded. (See Remark 1.5 (iii)) By the martingale property of $\{e^{-rt}u(S_t^{s,v},V_t^{v}),0\le t\le\tau_D\}$ we have

$$u(s,v)=\mathbb{E}e^{-r(\tau_D\wedge t)}u(S_{\tau_D\wedge t}^{s,v}).$$

Fix $(s,v) \in D^c$. For all $\psi \in C^{2,1}(\mathcal{O}) \cap \mathcal{W}$ such that $\psi \geq u$ and $\psi(s,v) = u(s,v)$. We have

$$\begin{aligned}
0 &= \mathbb{E}e^{-r(\tau_D \wedge t)} u(S^{s,v}_{\tau_D \wedge t}) - u(s,v) \\
&\leq \mathbb{E}e^{-r(\tau_D \wedge t)} \psi(S^{s,v}_{\tau_D \wedge t}) - \psi(s,v) \\
&= \mathbb{E}\bigg(\int_0^{\tau_D \wedge t} e^{-rq}(L\psi - r\psi)(S^{s,v}_q, V^v_q)\ \mathrm{d}q + M_{t \wedge \tau_D} \bigg) \\
&= \mathbb{E}\bigg(\int_0^{\tau_D \wedge t} e^{-rq}(L\psi - r\psi)(S^{s,v}_q, V^v_q)\ \mathrm{d}q \bigg)
\end{aligned}$$

where M is a martingale with $M_0 = 0$. Now observe that, by the right-continuity of (S,V), $\mathbb{P}((S^{s,v}_0, V^v_0) \in D^c, \tau_D > 0) = 1$, which implies $\lim_{t \to 0} \mathbb{P}((S^{s,v}_0, V^v_0) \in D^c, \tau_D > t) = 0$. Now we divide by t on both side and take limit as $t \to 0$, we get

$$L\psi(s,v) - r\psi(s,v) \geq 0$$

Note that $u(s,v) \geq g(s)$ by definition, so $u(s,v)$ is a subsolution. Supersolution property holds by a similar proof. □

Expectation estimates for Proposition 4.40

$$\begin{aligned}
&u(s,v,T) - u(s,v,T') \\
&\quad \leq C\bigg(\underbrace{\mathbb{E}\bigg[\frac{1}{2}\int_0^\delta V^v_q \mathrm{d}q\bigg]}_{(1)} + \underbrace{\mathbb{E}\bigg|\int_0^\delta \sqrt{V^v_q}\mathrm{d}W_q\bigg|}_{(2)} + \lambda\Phi(\rho)\delta + \underbrace{\rho\mathbb{E}Z_{\lambda\delta}}_{(3)} + \underbrace{\mathbb{E}|V^v_\delta - v|}_{(4)} + \underbrace{\mathbb{E}|V^v_\delta - v|^{\frac{1}{2}}}_{(5)} \bigg)
\end{aligned}$$

where $\delta = |T - T'|$. We estimate each underbraced term individually. We first make the observation that $\int_0^t e^{\lambda q}\mathrm{d}Z_{\lambda q} = \mu_1(e^{\lambda t} - 1)$ and $\mathbb{E}Z_t = t\mu_1$, where $\mu_1 = \mathbb{E}Z_1$.

(1)

$$\mathbb{E}\bigg[\int_0^\delta V_q \mathrm{d}q\bigg] = \int_0^\delta v + e^{-\lambda q}\bigg[\int_0^q \mu_1 \lambda e^{\lambda q'}\mathrm{d}q'\bigg]\mathrm{d}q = (v+\mu_1)\delta + \frac{1}{\lambda}(1 - e^{-\lambda\delta}) \leq (v+1+\mu_1)\delta$$

(2)

$$\mathbb{E}\bigg|\int_0^\delta \sqrt{V^v_q}\mathrm{d}W_q\bigg| \leq \bigg(\mathbb{E}\int_0^\delta V_q \mathrm{d}q\bigg)^{1/2} \leq v^{\frac{1}{2}}\delta^{\frac{1}{2}} + (1+\mu_1)^{\frac{1}{2}}\delta^{\frac{1}{2}}$$

The first inequality follows by Itô isometry and Cauchy-Schwarz inequality. The second inequality follows by (1) and the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a,b \geq 0$.

(3)
$$\rho \mathbb{E} Z_{\lambda\delta} = \rho\lambda\mu_1\delta,$$

(4)
$$\mathbb{E}|V_\delta^v - v| = \mathbb{E}\left| ve^{-\lambda\delta} - v + \int_0^\delta e^{-\lambda(\delta-q)} \mathrm{d}Z_{\lambda q}\right| \leq (v+\mu_1)(1-e^{-\lambda\delta}) = \lambda(v+\mu_1)\delta$$

(5)
$$\mathbb{E}|V_\delta^v - v|^{\frac{1}{2}} \leq (\lambda(v+\mu_1)\delta)^{\frac{1}{2}} \leq \lambda^{\frac{1}{2}} v^{\frac{1}{2}} \delta^{\frac{1}{2}} + \mu_1^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

The first inequality follows from Cauchy-Schwarz and (4). The second inequality is an application of $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$.

Adding these inequalities together,

$$u(s,v,T') - u(s,v,T) \leq C'(1 + v + v^{\frac{1}{2}})(\delta^{1/2} + \delta),$$

where C is a constant.

Proof of Lemma 4.31 Theorem 8.2 of [19] considers a problem of the form.

$$\begin{cases} -\tilde{L}f(s,v) + \gamma(s,v) = 0 & \text{for } (s,v) \in (s_1,s_2)\times(v_1,v_2) \equiv \Theta, \\ f(s,v_1) = \zeta(s) & \text{for } s \in [s_1,s_2] \\ f(s,v) = 0 & \text{for } (s,v) \in \{s_1,s_2\}\times[v_1,v_2). \end{cases} \tag{4.73}$$

This is a case of (4.31) on page 103 with $\psi(s,v) = 0$ with $(s,v) \in \{s_1,s_2\}\times[v_1,v_2)$. We want to show the last of these conditions can be replaced by a more general condition $f(s,v) = \psi(s,v)$. [19, Theorem 8.2] implies that there exists at most one viscosity solution to (4.73).

Let $\phi(s,v)$ be an arbitrary $C^{2,1}$ function, then by [19, Remark 2.7 (ii)], we have

$$\mathcal{P}^{2,+}(u-\phi)(s,v) = \{(p - \partial_v\phi, q - \partial_s\phi, A - \partial_{ss}\phi) : (p,q,A) \in \mathcal{P}^{2,+}u(s,v)\}.$$

For $(s,v) \in \Theta$, let $(p,q,A) \in \mathcal{P}^{2,+}f_1(s,v)$, then

$$-\frac{1}{2}s^2vA - (r + C_\rho)sq + \lambda vp + rf_1(s,v) + \gamma(s,v) \leq 0. \tag{4.74}$$

Moreover, we have

$$-\frac{1}{2}s^2v\partial_{ss}f_2(s,v) - (r+C_\rho)s\partial_s f_2(s,v) + \lambda v\partial_v f_2(s,v) + rf_2(s,v) + \gamma(s,v) = 0, \tag{4.75}$$

since f_2 is a classical solution of $-\tilde{L}f_2(s,v)+\gamma(s,v)=0$. If we subtract (4.75) from (4.74), we have

$$-\frac{1}{2}s^2v(A-\partial_{ss}f_2(s,v))-(r+C_\rho)s(q-\partial_s f_2(s,v))+\lambda v(p-\partial_v f_2(s,v))$$
$$+r(f_1(s,v)-f_2(s,v))\leq 0$$

This shows that f_1-f_2 is a viscosity subsolution of $\tilde{L}=0$. The viscosity supersolution property holds by an almost identical argument. Hence f_1-f_2 is a viscosity solution to the problem

$$\begin{cases} -\tilde{L}f(s,v)=0 & \text{for } (s,v)\in(s_1,s_2)\times(v_1,v_2)\equiv\Theta, \\ f(s,v_1)=0 & \text{for } s\in[s_1,s_2] \\ f(s,v)=0 & \text{for } (s,v)\in\{s_1,s_2\}\times[v_1,v_2). \end{cases} \tag{4.76}$$

This means f_1-f_2 is a viscosity solution (4.73) with $\zeta(s)=0$ and $\gamma(s,v)=0$. It is clear that the zero function satisfies (4.76) and this solution is unique by [19, Theorem 8.2]. This implies $f_1=f_2$.

Remark 4.52. It is clear that the proof of Lemma 4.31 can be extended to all (possibly degenerate) linear second order equations with non-negative characteristic form.

Sketch Proof of Proposition 4.46

Proof. By Remark 4.43 (3), for $(s,v,t)\in D^c$ we have

$$\frac{1}{2}s^2v\partial_{ss}\hat{u}(s,\hat{v},t)=-rs\partial_s u(s,v,t)+\partial_t\hat{u}(s,\hat{v},t)+ru(s,v,t)$$
$$-\int_0^\infty u(se^{-\rho z},v+z,t)-u(s,v,t)+s(1-e^{-\rho z})\partial_s u(s,v,t)\Pi(\mathrm{d}z),$$

where each of the derivatives exists. By the same convexity argument in Proposition 4.35, the integral term is positive. We have that

$$\frac{1}{2}s^2v\partial_{ss}\hat{u}(s,\hat{v},t)\leq\partial_t\hat{u}(s,\hat{v},t)+ru(s,v,t)+rs$$

For each value of $\hat{v}$, we consider $\hat{u}(s,\hat{v},t)$ as a function of two variables (s,t). By the same argument as the one in Proposition 4.35, we have

$$\lim_{D^c\ni(s',\hat{v},t')\to(s,\hat{v},t)}\partial_s\hat{u}(s,\hat{v},t)=-1.$$

Note, $\hat{v}$ is unchanged when taking the limit. □

Sketch Proof of Lemma 4.47

Proof. In the transformed continuation region, using the relationship between u and $\hat{u}$, we have that

$$\begin{aligned}\frac{1}{2}\hat{v}e^{\lambda t}\partial_{ss}u(s,\hat{v},t) =& ru(s,v,t) - r\partial_s u(s,v,t) + \partial_t u(s,\hat{v},t)\\ & \lambda\int_0^\infty u(se^{-\rho z}, v+z, t) - u(s,v,t) + s(1-e^{-\rho z})\partial_1 u(s,v,t)\Pi(\mathrm{d}z)\end{aligned}$$

This only difference between the right hand side this equation and the right hand side of the integral equation in Lemma 4.47 is that $-\lambda v\partial_v u(s,v)$ is replaced by $\partial_t u(s,\hat{v},t)$. The argument in Lemma 4.47 can be repeated exactly except we use the positivity of $\partial_t u(s,\hat{v},t)$ instead of positivity of $-\lambda v\partial_v u(s,v)$. □

Sketch Proof of Proposition 4.48

Proof. For a fixed $\hat{v}$, $\hat{b}(\hat{v},t) = b(\hat{v}e^{\lambda t},t)$ is monotone and left-continuous in t because $b(v,t)$ is jointly left-continuous in v and t.

If $\hat{b}(\hat{v},t)$ is discontinuous in t at t^0, then by Lemma 4.47, there exists s^*, t^* such that $\partial_{ss}\hat{u}(s,\hat{v},t) \geq C$ for $(s,v) \in (s^*, \hat{b}(\hat{v},t)) \times \{\hat{v}\} \times (t^0, t^*)$, where C is a constant. We just repeat the argument in Proposition 4.38. □

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